Notable Algebraic Topologists v.2, p.229, Edited by Bci2

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Algebraic Topology

Algebraic topology

Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism. In many situations this is too much to hope for and it is more prudent to aim for a more modest goal, classification up to homotopy equivalence.

Although algebraic topology primarily uses algebra to study topological problems, the converse, using topology to solve algebraic problems, is sometimes also possible. Algebraic topology, for example, allows for a convenient proof that any subgroup of a free group is again a free group.

The method of algebraic invariants

An older name for the subject was combinatorial topology, implying an emphasis on how a space X was constructed from simpler ones (the modern standard tool for such construction is the CW-complex). The basic method now applied in algebraic topology is to investigate spaces via algebraic invariants by mapping them, for example, to groups which have a great deal of manageable structure in a way that respects the relation of homeomorphism (or more general homotopy) of spaces. This allows one to recast statements about topological spaces into statements about groups, which are often easier to prove.

Two major ways in which this can be done are through → fundamental groups, or more generally → homotopy theory, and through homology and cohomology groups. The fundamental groups give us basic information about the structure of a topological space, but they are often nonabelian and can be difficult to work with. The fundamental group of a (finite) simplicial complex does have a finite presentation.

Homology and cohomology groups, on the other hand, are abelian and in many important cases finitely generated. Finitely generated abelian groups are completely classified and are particularly easy to work with.

Setting in category theory

In general, all constructions of algebraic topology are functorial; the notions of category, functor and natural transformation originated here. Fundamental groups and homology and cohomology groups are not only *invariants* of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the same associated groups, but their associated morphisms also correspond — a continuous mapping of spaces induces a group homomorphism on the associated groups, and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings.

Results on homology

Several useful results follow immediately from working with finitely generated abelian groups. The free rank of the n-th homology group of a simplicial complex is equal to the n-th Betti number, so one can use the homology groups of a simplicial complex to calculate its Euler-Poincaré characteristic. As another example, the top-dimensional integral homology group of a closed manifold detects orientability: this group is isomorphic to either the integers or 0, according as the manifold is orientable or not. Thus, a great deal of topological information is encoded in the homology of a given topological space.

Beyond simplicial homology, which is defined only for simplicial complexes, one can use the differential structure of smooth manifolds via de Rham cohomology, or Čech or sheaf cohomology to investigate the solvability of differential equations defined on the manifold in question. De Rham showed that all of these approaches were interrelated and that, for a closed, oriented manifold, the Betti numbers derived through simplicial homology were the same Betti numbers as those derived through de Rham cohomology. This was extended in the 1950s, when Eilenberg and Steenrod generalized this approach. They defined homology and cohomology as functors equipped with natural transformations subject to certain axioms (e.g., a weak equivalence of spaces passes to an isomorphism of homology groups), verified that all existing (co)homology theories satisfied these axioms, and then proved that such an axiomatization uniquely characterized the theory.

A new approach uses a functor from filtered spaces to crossed complexes defined directly and homotopically using relative homotopy groups; a higher homotopy van Kampen theorem proved for this functor enables basic results in algebraic topology, especially on the border between homology and homotopy, to be obtained without using singular homology or simplicial approximation. This approach is also called non abelian algebraic topology, and generalises to higher dimensions ideas coming from the fundamental group.

Applications of algebraic topology

Classic applications of algebraic topology include:

- The → Brouwer fixed point theorem: every continuous map from the unit *n*-disk to itself has a fixed point.
- The *n*-sphere admits a nowhere-vanishing continuous unit vector field if and only if *n* is odd. (For n=2, this is sometimes called the "hairy ball theorem".)
- The → Borsuk-Ulam theorem: any continuous map from the *n*-sphere to Euclidean *n*-space identifies at least one pair of antipodal points.
- Any subgroup of a free group is free. This result is quite interesting, because the statement is purely algebraic yet the simplest proof is topological. Namely, any free group G may be realized as the fundamental group of a graph X. The main theorem on covering spaces tells us that every subgroup H of G is the fundamental group of some covering space Y of X; but every such Y is again a graph. Therefore its fundamental group H is free.
- Topological combinatorics

Notable algebraic topologists

- → Frank Adams
- → Karol Borsuk
- → Luitzen Egbertus Jan Brouwer
- → William Browder
- → Nicolas Bourbaki
- → Henri Cartan
- → Otto Hermann Künneth
- → Samuel Eilenberg
- → Peter Freyd
- → Alexander Grothendieck
- Friedrich Hirzebruch
- → Heinz Hopf
- → Michael J. Hopkins
- → Witold Hurewicz
- Egbert van Kampen
- → Saunders Mac Lane
- → J.P. May

- → John Coleman Moore
- → Sergei Petrovich Novikov
- Lev Pontryagin
- → Daniel Quillen
- → Jean-Pierre Serre
- Norman Steenrod
- → Dennis Sullivan
- → René Thom
- → Hassler Whitney
- → J. H. C. Whitehead

Important theorems in algebraic topology

- → Borsuk-Ulam theorem
- → Brouwer fixed point theorem
- → Cellular approximation theorem
- → Eilenberg–Zilber theorem
- → Hurewicz theorem
- → Kunneth theorem
- → Poincaré duality theorem
- → Universal coefficient theorem
- → Van Kampen's theorem
- Generalized van Kampen's theorems [1]
- Higher homotopy, generalized van Kampen's theorem[2]
- → Whitehead's theorem

See also

- Important publications in algebraic topology
- GNUL Textbook on Algebraic Topology vol.1 [3][4]
- → Higher dimensional algebra
- → Higher category theory
- → Van Kampen's theorem
- → Groupoid
- → Lie groupoid
- → Lie algebroid
- → Grothendieck topology
- → Serre spectral sequence
- → Sheaf
- Homotopy
- → Homotopy theory
- → Fundamental group
- → Homology theory
- → Homological algebra
- → Cohomology theory
- → K-theory
- → Algebraic K-theory
- → TQFT

- Homotopy quantum field theory(HQFT)
- CW complex
- Simplicial complex
- Homology complex
- Algebroid
- Exact sequence

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[8] http://www.bangor.ac.uk/r.brown/hdaweb2.html

[9] http://www.bangor.ac.uk/~mas010/nonab-a-t.html

[10] http://www.bangor.ac.uk/~mas010/pdffiles/rbrsbookb-e090809.pdf

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[12] http://planetmath.org/?op=getobj&from=objects&id=5576

[13] http://www.math.uchicago.edu/~may/VIGRE/VIGREREU2008.html

[14] http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf

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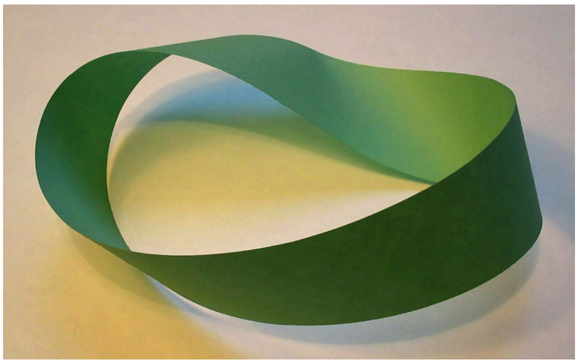
Topology

Topology (from the Greek τόπος, “place”, and λόγος, “study”) is a major area of mathematics concerned with spatial properties that are preserved under continuous deformations of objects, for example deformations that involve stretching, but no tearing or gluing. It emerged through the development of concepts from geometry and set theory, such as space, dimension, and transformation.



A Möbius strip, an object with only one surface and one edge. Such shapes are an object of study in topology.

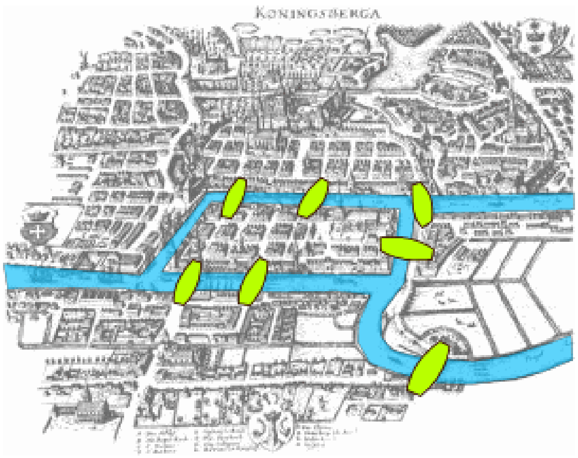
Ideas that are now classified as topological were expressed as early as 1736, and toward the end of the 19th century a distinct discipline developed, called in Latin the *geometria situs* (“geometry of place”) or *analysis situs* (Greek-Latin for “picking apart of place”), and later gaining the modern name of topology. In the middle of the 20th century, this was an important growth area within mathematics.

The word *topology* is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Of particular importance are *homeomorphisms*, which can be defined as continuous functions with a continuous inverse. For instance, the function $y = x^3$ is a homeomorphism of the real line.

Topology includes many subfields. The most basic and traditional division within topology is **point-set topology**, which establishes the foundational aspects of topology and investigates concepts inherent to topological spaces - basic examples being compactness and connectedness; → **algebraic topology**, which generally tries to measure degrees of connectivity using algebraic constructs such as homotopy groups and homology; and **geometric topology**, which primarily studies manifolds and their embeddings (placements) in other manifolds. Some of the most active areas, such as low dimensional topology and graph theory, do not fit neatly in this division.

See also: topology glossary for definitions of some of the terms used in topology and topological space for a more technical treatment of the subject.

History



The Seven Bridges of Königsberg is a famous problem solved by Euler.

Topology began with the investigation of certain questions in geometry. Euler's 1736 paper on *Seven Bridges of Königsberg* is regarded as one of the first topological results.

The term "Topologie" was introduced in German in 1847 by Johann Benedict Listing in *Vorstudien zur Topologie*, Vandenhoeck und Ruprecht, Göttingen, pp. 67, 1848, who had used the word for ten years in correspondence before its first appearance in print. "Topology," its English form, was introduced in 1883 in the journal *Nature* to distinguish "qualitative geometry from the ordinary geometry in which quantitative relations chiefly are treated". The term **topologist** in the sense of a specialist in topology was used in 1905 in the magazine *Spectator*. However, none of these uses corresponds exactly to the modern definition of topology.

Modern topology depends strongly on the ideas of set theory, developed by Georg Cantor in the later part of the 19th century. Cantor, in addition to setting down the basic ideas of set theory, considered point sets in Euclidean space, as part of his study of Fourier series.

Henri Poincaré published *Analysis Situs* in 1895, introducing the concepts of homotopy and homology, which are now considered part of → algebraic topology.

Maurice Fréchet, unifying the work on function spaces of Cantor, Volterra, Arzelà, Hadamard, Ascoli, and others, introduced the metric space in 1906. A metric space is now considered a special case of a general topological space. In 1914, Felix Hausdorff coined the term "topological space" and gave the definition for what is now called a Hausdorff space. In current usage, a topological space is a slight generalization of Hausdorff spaces, given in 1922 by Kazimierz Kuratowski.

For further developments, see point-set topology and → algebraic topology.

Elementary introduction

Topology, as a branch of mathematics can be formally defined as "the study of qualitative properties of certain objects (called topological spaces) that are invariant under certain kind of transformations (called continuous maps), especially those properties that are invariant under a certain kind of equivalence (called homeomorphism)."

The term *topology* is also used to refer to a structure imposed upon a set X, a structure which essentially 'characterizes' the set X as a topological space by taking proper care of properties such as convergence, connectedness and continuity, upon transformation.

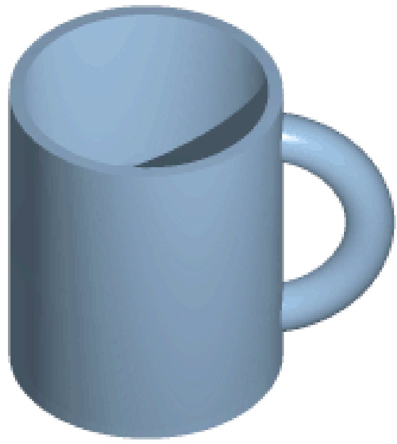
Topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics.

The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. For example, the square and the circle have many properties in common: they are both one dimensional objects (from a topological point of view) and both separate the plane into two parts, the part inside and the part outside.

One of the first papers in topology was the demonstration, by Leonhard Euler, that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This

result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges are connected to which islands or riverbanks. This problem, the *Seven Bridges of Königsberg*, is now a famous problem in introductory mathematics, and led to the branch of mathematics known as graph theory.

Similarly, the hairy ball theorem of algebraic topology says that "one cannot comb the hair flat on a hairy ball without creating a cowlick." This fact is immediately convincing to most people, even though they might not recognize the more formal statement of the theorem, that there is no nonvanishing continuous tangent vector field on the sphere. As with the *Bridges of Königsberg*, the result does not depend on the exact shape of the sphere; it applies to pear shapes and in fact any kind of smooth blob, as long as it has no holes.



A continuous deformation (homeomorphism) of a coffee cup into a doughnut (torus) and back.

In order to deal with these problems that do not rely on the exact shape of the objects, one must be clear about just what properties these problems *do* rely on. From this need arises the notion of homeomorphism. The impossibility of crossing each bridge just once applies to any arrangement of bridges homeomorphic to those in Königsberg, and the hairy ball theorem applies to any space homeomorphic to a sphere.

Intuitively two spaces are homeomorphic if one can be deformed into the other without cutting or gluing. A traditional joke is that a topologist can't distinguish a coffee mug from a doughnut, since a sufficiently pliable doughnut could be reshaped to the form of a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle. A precise definition of homeomorphic, involving a continuous function with a continuous inverse, is necessarily more technical.

Homeomorphism can be considered the most basic *topological equivalence*. Another is homotopy equivalence. This is harder to describe without getting technical, but the essential notion is that two objects are homotopy equivalent if they both result from "squishing" some larger object.

Equivalence classes of the English alphabet in uppercase sans-serif font (Myriad); left - homeomorphism, right - homotopy equivalence

{A, R} {B} {C, G, I, J, L, M, N, S, U, V, W, Z}
{D, O} {E, F, T, Y} {H, K} {P, Q} {X}

{A, R, D, O, P, Q} {B}
{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}

An introductory exercise is to classify the uppercase letters of the English alphabet according to homeomorphism and homotopy equivalence. The result depends partially on the font used. The figures use a sans-serif font named Myriad. Notice that homotopy equivalence is a rougher relationship than homeomorphism; a homotopy equivalence class can contain several of the homeomorphism classes. The simple case of homotopy equivalence described above can be used here to show two letters are homotopy equivalent, e.g. O fits inside P and the tail of the P can be squished to the "hole" part.

Thus, the homeomorphism classes are: one hole two tails, two holes no tail, no holes, one hole no tail, no holes three tails, a bar with four tails (the "bar" on the *K* is almost too short to see), one hole one tail, and no holes four tails.

The homotopy classes are larger, because the tails can be squished down to a point. The homotopy classes are: one hole, two holes, and no holes.

To be sure we have classified the letters correctly, we not only need to show that two letters in the same class are equivalent, but that two letters in different classes are not equivalent. In the case of homeomorphism, this can be done by suitably selecting points and showing their removal disconnects the letters differently. For example, X and Y are not homeomorphic because removing the center point of the X leaves four pieces; whatever point in Y corresponds to this point, its removal can leave at most three pieces. The case of homotopy equivalence is harder and requires a more elaborate argument showing an algebraic invariant, such as the → fundamental group, is different on the supposedly differing classes.

Letter topology has some practical relevance in stencil typography. The font Braggadocio, for instance, has stencils that are made of one connected piece of material.

Mathematical definition

Let **X** be any set and let ***T*** be a family of subsets of **X**. Then ***T*** is a **topology** on **X** iff

1. Both the empty set and **X** are elements of ***T***.
2. Any union of arbitrarily many elements of ***T*** is an element of ***T***.
3. Any intersection of finitely many elements of ***T*** is an element of ***T***.

If ***T*** is a topology on **X**, then the pair (**X**, ***T***) is called a **topological space**, and the notation $\mathbf{X}_{\boldsymbol{T}}$ is used to denote a set **X** endowed with the particular topology ***T***.

The **open** sets in X are defined to be the members of T; note that in general not all subsets of **X** need be in ***T***. A subset of **X** is said to be closed if its complement is in ***T*** (i.e., it is open). A subset of **X** may be open, closed, both, or neither.

A function or map from one topological space to another is called **continuous** if the inverse image of any open set is open. If the function maps the real numbers to the real numbers (both spaces with the Standard Topology), then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. Another way of saying this is that the function has a natural extension to the topology. If two spaces are homeomorphic, they have identical topological properties, and are considered to be topologically the same. The cube and the sphere are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut.

Topology topics

Some theorems in general topology

- Every closed interval in **R** of finite length is compact. More is true: In $\mathbf{R}^n$, a set is compact if and only if it is closed and bounded. (See Heine-Borel theorem).
- Every continuous image of a compact space is compact.
- Tychonoff's theorem: The (arbitrary) product of compact spaces is compact.
- A compact subspace of a Hausdorff space is closed.
- Every continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism.
- Every sequence of points in a compact metric space has a convergent subsequence.
- Every interval in **R** is connected.
- Every compact m-manifold can be embedded in some Euclidean space $\mathbf{R}^n$.

- The continuous image of a connected space is connected.
- A metric space is Hausdorff, also normal and paracompact.
- The metrization theorems provide necessary and sufficient conditions for a topology to come from a metric.
- The Tietze extension theorem: In a normal space, every continuous real-valued function defined on a closed subspace can be extended to a continuous map defined on the whole space.
- Any open subspace of a Baire space is itself a Baire space.
- The Baire category theorem: If X is a complete metric space or a locally compact Hausdorff space, then the interior of every union of countably many nowhere dense sets is empty.
- On a paracompact Hausdorff space every open cover admits a partition of unity subordinate to the cover.
- Every path-connected, locally path-connected and semi-locally simply connected space has a universal cover.

General topology also has some surprising connections to other areas of mathematics. For example:

- In number theory, Furstenberg's proof of the infinitude of primes.

Some useful notions from algebraic topology

See also list of algebraic topology topics.

- Homology and cohomology: Betti numbers, Euler characteristic, degree of a continuous mapping.
- Operations: cup product, Massey product
- Intuitively-attractive applications: Brouwer fixed-point theorem, Hairy ball theorem, → Borsuk-Ulam theorem, Ham sandwich theorem.
- Homotopy groups (including the → fundamental group).
- Chern classes, Stiefel-Whitney classes, Pontryagin classes.

Generalizations

Occasionally, one needs to use the tools of topology but a "set of points" is not available. In pointless topology one considers instead the lattice of open sets as the basic notion of the theory, while → Grothendieck topologies are certain structures defined on arbitrary categories which allow the definition of sheaves on those categories, and with that the definition of quite general cohomology theories.

Topology in art and literature

- Some M. C. Escher works illustrate topological concepts, such as Möbius strips and non-orientable spaces.

See also

- Topology glossary
- List of topology topics
- List of general topology topics
- List of geometric topology topics
- List of algebraic topology topics
- Publications in topology

Further reading

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External links

- Elementary Topology: A First Course [1] Viro, Ivanov, Netsvetaev, Kharlamov
- ODP category [2]
- The Topological Zoo [3] at The Geometry Center
- Topology Atlas [4]
- Topology Course Lecture Notes [5] Aisling McCluskey and Brian McMaster, Topology Atlas
- Topology Glossary [6]
- Moscow 1935: Topology moving towards America [7], a historical essay by → Hassler Whitney.

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[5] http://at.yorku.ca/i/a/a/b/23.htm
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[7] http://www.ams.org/online_bks/hmath1/hmath1-whitney10.pdf

Glossary of topology

This is a glossary of some terms used in the branch of mathematics known as → topology. Although there is no absolute distinction between different areas of topology, the focus here is on general topology. The following definitions are also fundamental to → algebraic topology, differential topology and geometric topology.

See the article on topological spaces for basic definitions and examples, and see the article on → topology for a brief history and description of the subject area. See Naive set theory, Axiomatic set theory, and Function for definitions concerning sets and functions. The following articles may also be useful. These either contain specialised vocabulary within general topology or provide more detailed expositions of the definitions given below. The list of general topology topics and the list of examples in general topology will also be very helpful.

- Compact space
- Connected space
- Continuity
- Metric space
- Separated sets
- Separation axiom
- Topological space
- Uniform space

All spaces in this glossary are assumed to be topological spaces unless stated otherwise.

Contents

A

Accessible

See T_1.

Accumulation point

See limit point.

Alexandrov topology

A space *X* has the Alexandrov topology (or is **finitely generated**) if arbitrary intersections of open sets in *X* are open, or equivalently, if arbitrary unions of closed sets are closed.

Almost discrete

A space is almost discrete if every open set is closed (hence clopen). The almost discrete spaces are precisely the finitely generated zero-dimensional spaces.

Approach space

An approach space is a generalization of metric space based on point-to-set distances, instead of point-to-point.

B

Baire space

This has two distinct common meanings:

1. A space is a **Baire space** if the intersection of any countable collection of dense open sets is dense; see Baire space.
2. **Baire space** is the set of all functions from the natural numbers to the natural numbers, with the topology of pointwise convergence; see Baire space (set theory).

Base

A collection *B* of open sets is a base (or **basis**) for a topology τ if every open set in τ is a union of sets in B. The topology τ is the smallest topology on X containing B and is said to be generated by B.

Basis

See **Base**.

Borel algebra

The Borel algebra on a topological space (X, τ) is the smallest σ -algebra containing all the open sets. It is obtained by taking intersection of all σ -algebras on X containing τ.

Borel set

A Borel set is an element of a Borel algebra.

Boundary

The boundary (or **frontier**) of a set is the set's closure minus its interior. Equivalently, the boundary of a set is the intersection of its closure with the closure of its complement. Boundary of a set A is denoted by ∂A

Bounded

A set in a metric space is bounded if it has finite diameter. Equivalently, a set is bounded if it is contained in some open ball of finite radius. A function taking values in a metric space is bounded if its image is a bounded set.

C

Category of topological spaces

The category **Top** has topological spaces as objects and continuous maps as morphisms.

Cauchy sequence

A sequence $\{x_n\}$ in a metric space (M, d) is a Cauchy sequence if, for every positive real number r, there is an integer N such that for all integers $m, n > N$, we have $d(x_m, x_n) < r$.

Clopen set

A set is clopen if it is both open and closed.

Closed ball

If (M, d) is a metric space, a closed ball is a set of the form $D(x; r) := \{y \text{ in } M : d(x, y) \leq r\}$, where x is in M and r is a positive real number, the **radius** of the ball. A closed ball of radius r is a **closed *r*-ball**. Every closed ball is a closed set in the topology induced on M by d. Note that the closed ball $D(x; r)$ might not be equal to the closure of the open ball $B(x; r)$.

Closed set

A set is closed if its complement is a member of the topology.

Closed function

A function from one space to another is closed if the image of every closed set is closed.

Closure

The closure of a set is the smallest closed set containing the original set. It is equal to the intersection of all closed sets which contain it. An element of the closure of a set *S* is a **point of closure** of *S*.

Closure operator

See **Kuratowski closure axioms**.

Coarser topology

If *X* is a set, and if T_1 and T_2 are topologies on *X*, then T_1 is coarser (or **smaller**, **weaker**) than T_2 if T_1 is contained in T_2. Beware, some authors, especially analysts, use the term **stronger**.

Comeagre

A subset *A* of a space *X* is **comeagre** (**comeager**) if its complement $X\backslash A$ is meagre. Also called **residual**.

Compact

A space is compact if every open cover has a finite subcover. Every compact space is Lindelöf and paracompact. Therefore, every compact Hausdorff space is normal. See also **quasicompact**.

Compact-open topology

The compact-open topology on the set $C(X, Y)$ of all continuous maps between two spaces *X* and *Y* is defined as follows: given a compact subset *K* of *X* and an open subset *U* of *Y*, let $V(K, U)$ denote the set of all maps *f* in $C(X, Y)$ such that $f(K)$ is contained in *U*. Then the collection of all such $V(K, U)$ is a subbase for the compact-open topology.

Complete

A metric space is complete if every Cauchy sequence converges.

Completely metrizable/completely metrisable

See **complete space**.

Completely normal

A space is completely normal if any two separated sets have disjoint neighbourhoods.

Completely normal Hausdorff

A completely normal Hausdorff space (or $\mathbf{T_5}$ space) is a completely normal T_1 space. (A completely normal space is Hausdorff if and only if it is T_1, so the terminology is consistent.) Every completely normal Hausdorff space is normal Hausdorff.

Completely regular

A space is completely regular if, whenever *C* is a closed set and *x* is a point not in *C*, then *C* and $\{x\}$ are functionally separated.

Completely T_3

See **Tychonoff**.

Component

See **Connected component/Path-connected component**.

Connected

A space is connected if it is not the union of a pair of disjoint nonempty open sets. Equivalently, a space is connected if the only clopen sets are the whole space and the empty set.

Connected component

A connected component of a space is a maximal nonempty connected subspace. Each connected component is closed, and the set of connected components of a space is a partition of that space.

Continuous

A function from one space to another is continuous if the preimage of every open set is open.

Contractible

A space *X* is contractible if the identity map on *X* is homotopic to a constant map. Every contractible space is simply connected.

Coproduct topology

If $\{X_i\}$ is a collection of spaces and *X* is the (set-theoretic) disjoint union of $\{X_i\}$, then the coproduct topology (or **disjoint union topology**, **topological sum** of the X_i) on *X* is the finest topology for which all the injection maps are continuous.

Countably compact

A space is countably compact if every countable open cover has a finite subcover. Every countably compact space is pseudocompact and weakly countably compact.

Countably locally finite

A collection of subsets of a space *X* is **countably locally finite** (or **σ-locally finite**) if it is the union of a countable collection of locally finite collections of subsets of *X*.

Cover

A collection of subsets of a space is a cover (or **covering**) of that space if the union of the collection is the whole space.

Covering

See **Cover**.

Cut point

If *X* is a connected space with more than one point, then a point *x* of *X* is a cut point if the subspace $X - \{x\}$ is disconnected.

D

Dense set

A set is dense if it has nonempty intersection with every nonempty open set. Equivalently, a set is dense if its closure is the whole space.

Derived set

If *X* is a space and *S* is a subset of *X*, the derived set of *S* in *X* is the set of limit points of *S* in *X*.

Diameter

If (M, d) is a metric space and *S* is a subset of *M*, the diameter of *S* is the supremum of the distances $d(x, y)$, where *x* and *y* range over *S*.

Discrete metric

The discrete metric on a set *X* is the function $d : X \times X \rightarrow \mathbf{R}$ such that for all *x*, *y* in *X*, $d(x, x) = 0$ and $d(x, y) = 1$ if $x \neq y$. The discrete metric induces the discrete topology on *X*.

Discrete space

A space *X* is discrete if every subset of *X* is open. We say that *X* carries the **discrete topology**.

Discrete topology

See **discrete space**.

Disjoint union topology

See **Coproduct topology**.

Dispersion point

If X is a connected space with more than one point, then a point x of X is a dispersion point if the subspace $X - \{x\}$ is hereditarily disconnected (its only connected components are the one-point sets).

Distance

See **metric space**.

Dunce hat (topology)

E

Entourage

See **Uniform space**.

Exterior

The exterior of a set is the interior of its complement.

F

F_σ set

An F_σ set is a countable union of closed sets.

Filter

A filter on a space X is a nonempty family F of subsets of X such that the following conditions hold:

1. The empty set is not in F.
2. The intersection of any finite number of elements of F is again in F.
3. If A is in F and if B contains A, then B is in F.

Finer topology

If X is a set, and if T_1 and T_2 are topologies on X, then T_2 is finer (or **larger**, **stronger**) than T_1 if T_2 contains T_1. Beware, some authors, especially analysts, use the term **weaker**.

Finitely generated

See **Alexandrov topology**.

First category

See **Meagre**.

First-countable

A space is first-countable if every point has a countable local base.

Fréchet

See $\mathbf{T_1}$.

Frontier

See **Boundary**.

Full set

A compact subset K of the complex plane is called **full** if its complement is connected. For example, the closed unit disk is full, while the unit circle is not.

Functionally separated

Two sets A and B in a space X are functionally separated if there is a continuous map f: $X \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

G

***G_δ* set**

A G_δ set is a countable intersection of open sets.

H

Hausdorff

A Hausdorff space (or **T_2 space**) is one in which every two distinct points have disjoint neighbourhoods. Every Hausdorff space is T_1.

Hereditary

A property of spaces is said to be hereditary if whenever a space has that property, then so does every subspace of it. For example, second-countability is a hereditary property.

Homeomorphism

If X and Y are spaces, a homeomorphism from X to Y is a bijective function $f : X \to Y$ such that f and f^{-1} are continuous. The spaces X and Y are then said to be **homeomorphic**. From the standpoint of topology, homeomorphic spaces are identical.

Homogeneous

A space X is homogeneous if, for every x and y in X, there is a homeomorphism $f : X \rightarrow X$ such that $f(x) = y$. Intuitively, the space looks the same at every point. Every topological group is homogeneous.

Homotopic maps

Two continuous maps f, $g : X \rightarrow Y$ are homotopic (in Y) if there is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all x in X. Here, $X \times [0, 1]$ is given the product topology. The function H is called a **homotopy** (in Y) between f and g.

Homotopy

See **Homotopic maps**.

Hyper-connected

A space is hyper-connected if no two non-empty open sets are disjoint. Every hyper-connected space is connected.

I

Identification map

See **Quotient map**.

Identification space

See **Quotient space**.

Indiscrete space

See **Trivial topology**.

Infinite-dimensional topology

See **Hilbert manifods** and **Q-manifolds**, i.e. (generalized) manifolds modelled on the Hilbert space and on the Hilbert cube respectively.

Interior

The interior of a set is the largest open set contained in the original set. It is equal to the union of all open sets contained in it. An element of the interior of a set S is an **interior point** of S.

Interior point

See **Interior**.

Isolated point

A point x is an isolated point if the singleton $\{x\}$ is open. More generally, if S is a subset of a space X, and if x is a point of S, then x is an isolated point of S if $\{x\}$ is open in the subspace topology on S.

Isometric isomorphism

If M_1 and M_2 are metric spaces, an isometric isomorphism from M_1 to M_2 is a bijective isometry $f : M_1 \to M_2$. The metric spaces are then said to be **isometrically isomorphic**. From the standpoint of metric space theory, isometrically isomorphic spaces are identical.

Isometry

If (M_1, d_1) and (M_2, d_2) are metric spaces, an isometry from M_1 to M_2 is a function $f : M_1 \to M_2$ such that $d_2(f(x), f(y)) = d_1(x, y)$ for all x, y in M_1. Every isometry is injective, although not every isometry is surjective.

K

Kolmogorov axiom

See $\mathbf{T_0}$.

Kuratowski closure axioms

The Kuratowski closure axioms is a set of axioms satisfied by the function which takes each subset of X to its closure:

1. *Isotonicity*: Every set is contained in its closure.
2. *Idempotence*: The closure of the closure of a set is equal to the closure of that set.
3. *Preservation of binary unions*: The closure of the union of two sets is the union of their closures.
4. *Preservation of nullary unions*: The closure of the empty set is empty.

If c is a function from the power set of X to itself, then c is a **closure operator** if it satisfies the Kuratowski closure axioms. The Kuratowski closure axioms can then be used to define a topology on X by declaring the closed sets to be the fixed points of this operator, i.e. a set A is closed if and only if $c(A) = A$.

L

Larger topology

See **Finer topology**.

Limit point

A point x in a space X is a limit point of a subset S if every open set containing x also contains a point of S other than x itself. This is equivalent to requiring that every neighbourhood of x contains a point of S other than x itself.

Limit point compact

See **Weakly countably compact**.

Lindelöf

A space is Lindelöf if every open cover has a countable subcover.

Local base

A set B of neighbourhoods of a point x of a space X is a local base (or **local basis**, **neighbourhood base**, **neighbourhood basis**) at x if every neighbourhood of x contains some member of B.

Local basis

See **Local base**.

Locally closed subset

A subset of a topological space that is the intersection of an open and a closed subset. Equivalently, it is a relatively open subset of its closure.

Locally compact

A space is locally compact if every point has a local base consisting of compact neighbourhoods. Every locally compact Hausdorff space is Tychonoff.

Locally connected

A space is locally connected if every point has a local base consisting of connected neighbourhoods.

Locally finite

A collection of subsets of a space is locally finite if every point has a neighbourhood which has nonempty intersection with only finitely many of the subsets. See also **countably locally finite**, **point finite**.

Locally metrizable/Locally metrisable

A space is locally metrizable if every point has a metrizable neighbourhood.

Locally path-connected

A space is locally path-connected if every point has a local base consisting of path-connected neighbourhoods. A locally path-connected space is connected if and only if it is path-connected.

Locally simply connected

A space is locally simply connected if every point has a local base consisting of simply connected neighbourhoods.

Loop

If x is a point in a space X, a loop at x in X (or a loop in X with basepoint x) is a path f in X, such that $f(0) = f(1) = x$. Equivalently, a loop in X is a continuous map from the unit circle S^1 into X.

M

Meagre

If X is a space and A is a subset of X, then A is meagre in X (or of **first category** in X) if it is the countable union of nowhere dense sets. If A is not meagre in X, A is of **second category** in X.

Metric

See **Metric space**.

Metric invariant

A metric invariant is a property which is preserved under isometric isomorphism.

Metric map

If X and Y are metric spaces with metrics d_X and d_Y respectively, then a metric map is a function f from X to Y, such that for any points x and y in X, $d_Y(f(x), f(y)) \leq d_X(x, y)$. A metric map is strictly metric if the above inequality is strict for all x and y in X.

Metric space

A metric space (M, d) is a set M equipped with a function $d : M \times M \to \mathbf{R}$ satisfying the following axioms for all x, y, and z in M:

1. $d(x, y) \geq 0$
2. $d(x, x) = 0$
3. if $d(x, y) = 0$ then $x = y$ (*identity of indiscernibles*)
4. $d(x, y) = d(y, x)$ (*symmetry*)
5. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*)

The function d is a **metric** on M, and $d(x, y)$ is the **distance** between x and y. The collection of all open balls of M is a base for a topology on M; this is the topology on M induced by d. Every metric space is Hausdorff and paracompact (and hence normal and Tychonoff). Every metric space is first-countable.

Metrizable/Metrisable

A space is metrizable if it is homeomorphic to a metric space. Every metrizable space is Hausdorff and paracompact (and hence normal and Tychonoff). Every metrizable space is first-countable.

Monolith

Every non-empty ultra-connected compact space X has a largest proper open subset; this subset is called a **monolith**.

N

Neighbourhood/Neighborhood

A neighbourhood of a point x is a set containing an open set which in turn contains the point x. More generally, a neighbourhood of a set S is a set containing an open set which in turn contains the set S. A neighbourhood of a point x is thus a neighbourhood of the singleton set $\{x\}$. (Note that under this definition, the neighbourhood itself need not be open. Many authors require that neighbourhoods be open; be careful to note conventions.)

Neighbourhood base/basis

See **Local base**.

Neighbourhood system for a point *x*

A neighbourhood system at a point x in a space is the collection of all neighbourhoods of x.

Net

A net in a space X is a map from a directed set A to X. A net from A to X is usually denoted (x_α), where α is an index variable ranging over A. Every sequence is a net, taking A to be the directed set of natural numbers with the usual ordering.

Normal

A space is normal if any two disjoint closed sets have disjoint neighbourhoods. Every normal space admits a partition of unity.

Normal Hausdorff

A normal Hausdorff space (or $\mathbf{T_4}$ space) is a normal T_1 space. (A normal space is Hausdorff if and only if it is T_1, so the terminology is consistent.) Every normal Hausdorff space is Tychonoff.

Nowhere dense

A nowhere dense set is a set whose closure has empty interior.

O

Open cover

An open cover is a cover consisting of open sets.

Open ball

If (M, d) is a metric space, an open ball is a set of the form $B(x; r) := \{y \text{ in } M : d(x, y) < r\}$, where x is in M and r is a positive real number, the **radius** of the ball. An open ball of radius r is an **open *r*-ball**. Every open ball is an open set in the topology on M induced by d.

Open condition

See **open property**.

Open set

An open set is a member of the topology.

Open function

A function from one space to another is open if the image of every open set is open.

Open property

A property of points in a topological space is said to be "open" if those points which possess it form an open set. Such conditions often take a common form, and that form can be said to be an *open condition*; for example, in metric spaces, one defines an open ball as above, and says that "strict inequality is an open condition".

P

Paracompact

A space is paracompact if every open cover has a locally finite open refinement. Paracompact Hausdorff spaces are normal.

Partition of unity

A partition of unity of a space X is a set of continuous functions from X to $[0, 1]$ such that any point has a neighbourhood where all but a finite number of the functions are identically zero, and the sum of all the functions on the entire space is identically 1.

Path

A path in a space X is a continuous map f from the closed unit interval $[0, 1]$ into X. The point $f(0)$ is the initial point of f; the point $f(1)$ is the terminal point of f.

Path-connected

A space X is path-connected if, for every two points x, y in X, there is a path f from x to y, i.e., a path with initial point $f(0) = x$ and terminal point $f(1) = y$. Every path-connected space is connected.

Path-connected component

A path-connected component of a space is a maximal nonempty path-connected subspace. The set of path-connected components of a space is a partition of that space, which is finer than the partition into connected components. The set of path-connected components of a space X is denoted $\pi_0(X)$.

Point

A point is an element of a topological space. More generally, a point is an element of any set with an underlying topological structure; e.g. an element of a metric space or a topological group is also a "point".

Point of closure

See **Closure**.

Polish

A space is Polish if it is separable and topologically complete, i.e. if it is homeomorphic to a separable and complete metric space.

Pre-compact

See **Relatively compact**.

Product topology

If $\{X_i\}$ is a collection of spaces and X is the (set-theoretic) product of $\{X_i\}$, then the product topology on X is the coarsest topology for which all the projection maps are continuous.

Proper function/mapping

A continuous function f from a space X to a space Y is proper if $f^{-1}(C)$ is a compact set in X for any compact subspace C of Y.

Proximity space

A proximity space $(X, \boldsymbol{\delta})$ is a set X equipped with a binary relation $\boldsymbol{\delta}$ between subsets of X satisfying the following properties:

For all subsets A, B and C of X,

1. $A\ \boldsymbol{\delta}\ B$ implies $B\ \boldsymbol{\delta}\ A$
2. $A\ \boldsymbol{\delta}\ B$ implies A is non-empty
3. If A and B have non-empty intersection, then $A\ \boldsymbol{\delta}\ B$
4. $A\ \boldsymbol{\delta}\ (B \cup C)$ iff ($A\ \boldsymbol{\delta}\ B$ or $A\ \boldsymbol{\delta}\ C$)
5. If, for all subsets E of X, we have ($A\ \boldsymbol{\delta}\ E$ or $B\ \boldsymbol{\delta}\ E$), then we must have $A\ \boldsymbol{\delta}\ (X - B)$

Pseudocompact

A space is pseudocompact if every real-valued continuous function on the space is bounded.

Pseudometric

See **Pseudometric space**.

Pseudometric space

A pseudometric space (M, d) is a set M equipped with a function $d : M \times M \to \mathbf{R}$ satisfying all the conditions of a metric space, except possibly the identity of indiscernibles. That is, points in a pseudometric space may be "infinitely close" without being identical. The function d is a **pseudometric** on M. Every metric is a pseudometric.

Punctured neighbourhood/Punctured neighborhood

A punctured neighbourhood of a point x is a neighbourhood of x, minus $\{x\}$. For instance, the interval $(-1, 1) = \{y : -1 < y < 1\}$ is a neighbourhood of $x = 0$ in the real line, so the set $(-1, 0) \cup (0, 1) = (-1, 1) - \{0\}$ is a punctured neighbourhood of 0.

Q

Quasicompact

See **compact**. Some authors define "compact" to include the Hausdorff separation axiom, and they use the term **quasicompact** to mean what we call in this glossary simply "compact" (without the Hausdorff axiom). This convention is most commonly found in French, and branches of mathematics heavily influenced by the French.

Quotient map

If X and Y are spaces, and if f is a surjection from X to Y, then f is a quotient map (or **identification map**) if, for every subset U of Y, U is open in Y if and only if $f^{-1}(U)$ is open in X. In other words, Y has the f-strong topology. Equivalently, f is a quotient map if and only if it is the transfinite composition of maps $X \to X/Z$, where $Z \subset X$ is a subset. Note that this doesn't imply that f is an open function.

Quotient space

If X is a space, Y is a set, and $f : X \to Y$ is any surjective function, then the quotient topology on Y induced by f is the finest topology for which f is continuous. The space X is a quotient space or **identification space**. By definition, f is a quotient map. The most common example of this is to consider an equivalence relation on X, with Y the set of equivalence classes and f the natural projection map. This construction is dual to the construction of the subspace topology.

R

Refinement

A cover K is a refinement of a cover L if every member of K is a subset of some member of L.

Regular

A space is regular if, whenever C is a closed set and x is a point not in C, then C and x have disjoint neighbourhoods.

Regular Hausdorff

A space is regular Hausdorff (or $\mathbf{T_3}$) if it is a regular T_0 space. (A regular space is Hausdorff if and only if it is T_0, so the terminology is consistent.)

Regular open

An open subset U of a space X is regular open if it equals the interior of its closure. An example of a non-regular open set is the set $U = (0, 1) \cup (1, 2)$ in $\mathbf{R}$ with its normal topology, since 1 is in the interior of the closure of U, but not in U. The regular open subsets of a space form a complete Boolean algebra.

Relatively compact

A subset Y of a space X is relatively compact in X if the closure of Y in X is compact.

Residual

If X is a space and A is a subset of X, then A is residual in X if the complement of A is meagre in X. Also called **comeagre** or **comeager**.

S

Second category

See **Meagre**.

Second-countable

A space is second-countable if it has a countable base for its topology. Every second-countable space is first-countable, separable, and Lindelöf.

Semilocally simply connected

A space *X* is semilocally simply connected if, for every point *x* in *X*, there is a neighbourhood *U* of *x* such that every loop at *x* in *U* is homotopic in *X* to the constant loop *x*. Every simply connected space and every locally simply connected space is semilocally simply connected. (Compare with locally simply connected; here, the homotopy is allowed to live in *X*, whereas in the definition of locally simply connected, the homotopy must live in *U*.)

Separable

A space is separable if it has a countable dense subset.

Separated

Two sets *A* and *B* are separated if each is disjoint from the other's closure.

Sequentially compact

A space is sequentially compact if every sequence has a convergent subsequence. Every sequentially compact space is countably compact, and every first-countable, countably compact space is sequentially compact.

Short map

See **metric map**

Simply connected

A space is simply connected if it is path-connected and every loop is homotopic to a constant map.

Smaller topology

See **Coarser topology**.

f **-Strong topology**

Let $f: X \to Y$ be a map of topological spaces. We say that Y has the f-strong topology if, for every subset $U \subset Y$, one has that U is open in Y if and only if $f^{-1}(U)$ is open in X

Stronger topology

See **Finer topology**. Beware, some authors, especially analysts, use the term **weaker topology**.

Subbase

A collection of open sets is a subbase (or **subbasis**) for a topology if every non-empty proper open set in the topology is a union of finite intersections of sets in the subbase. If *B* is *any* collection of subsets of a set *X*, the topology on *X* generated by *B* is the smallest topology containing *B*; this topology consists of the empty set, *X* and all unions of finite intersections of elements of *B*.

Subbasis

See **Subbase**.

Subcover

A cover *K* is a subcover (or **subcovering**) of a cover *L* if every member of *K* is a member of *L*.

Subcovering

See **Subcover**.

Subspace

If T is a topology on a space X, and if A is a subset of X, then the subspace topology on A induced by T consists of all intersections of open sets in T with A. This construction is dual to the construction of the quotient topology.

T

T_0

A space is T_0 (or **Kolmogorov**) if for every pair of distinct points x and y in the space, either there is an open set containing x but not y, or there is an open set containing y but not x.

T_1

A space is T_1 (or **Fréchet** or **accessible**) if for every pair of distinct points x and y in the space, there is an open set containing x but not y. (Compare with T_0; here, we are allowed to specify which point will be contained in the open set.) Equivalently, a space is T_1 if all its singletons are closed. Every T_1 space is T_0.

T_2

See **Hausdorff space**.

T_3

See **Regular Hausdorff**.

$T_{3.5}$

See **Tychonoff space**.

T_4

See **Normal Hausdorff**.

T_5

See **Completely normal Hausdorff**.

Top

See **Category of topological spaces**.

Topological invariant

A topological invariant is a property which is preserved under homeomorphism. For example, compactness and connectedness are topological properties, whereas boundedness and completeness are not. → Algebraic topology is the study of topologically invariant abstract algebra constructions on topological spaces.

Topological space

A topological space (X, T) is a set X equipped with a collection T of subsets of X satisfying the following axioms:

1. The empty set and X are in T.
2. The union of any collection of sets in T is also in T.
3. The intersection of any pair of sets in T is also in T.

The collection T is a **topology** on X.

Topological sum

See **Coproduct topology**.

Topologically complete

A space is topologically complete if it is homeomorphic to a complete metric space.

Topology

See **Topological space**.

Totally bounded

A metric space *M* is totally bounded if, for every $r > 0$, there exist a finite cover of *M* by open balls of radius *r*. A metric space is compact if and only if it is complete and totally bounded.

Totally disconnected

A space is totally disconnected if it has no connected subset with more than one point.

Trivial topology

The trivial topology (or **indiscrete topology**) on a set *X* consists of precisely the empty set and the entire space *X*.

Tychonoff

A Tychonoff space (or **completely regular Hausdorff** space, **completely** $\mathbf{T_3}$ space, $\mathbf{T_{3.5}}$ space) is a completely regular T_0 space. (A completely regular space is Hausdorff if and only if it is T_0, so the terminology is consistent.) Every Tychonoff space is regular Hausdorff.

U

Ultra-connected

A space is ultra-connected if no two non-empty closed sets are disjoint. Every ultra-connected space is path-connected.

Ultrametric

A metric is an ultrametric if it satisfies the following stronger version of the triangle inequality: for all x, y, z in M, $d(x, z) \le \max(d(x, y), d(y, z))$.

Uniform isomorphism

If *X* and *Y* are uniform spaces, a uniform isomorphism from *X* to *Y* is a bijective function $f : X \to Y$ such that f and f^{-1} are uniformly continuous. The spaces are then said to be uniformly isomorphic and share the same uniform properties.

Uniformizable/Uniformisable

A space is uniformizable if it is homeomorphic to a uniform space.

Uniform space

A uniform space is a set *U* equipped with a nonempty collection Φ of subsets of the Cartesian product $X \times X$ satisfying the following axioms:

1. if U is in Φ, then U contains $\{ (x, x) \mid x \text{ in } X \}$.
2. if U is in Φ, then $\{ (y, x) \mid (x, y) \text{ in } U \}$ is also in Φ
3. if U is in Φ and V is a subset of $X \times X$ which contains U, then V is in Φ
4. if U and V are in Φ, then $U \cap V$ is in Φ
5. if U is in Φ, then there exists V in Φ such that, whenever (x, y) and (y, z) are in V, then (x, z) is in U.

The elements of Φ are called **entourages**, and Φ itself is called a **uniform structure** on *U*.

Uniform structure

See **Uniform space**.

W

Weak topology

The weak topology on a set, with respect to a collection of functions from that set into topological spaces, is the coarsest topology on the set which makes all the functions continuous.

Weaker topology

See **Coarser topology**. Beware, some authors, especially analysts, use the term **stronger topology**.

Weakly countably compact

A space is weakly countably compact (or **limit point compact**) if every infinite subset has a limit point.

Weakly hereditary

A property of spaces is said to be weakly hereditary if whenever a space has that property, then so does every closed subspace of it. For example, compactness and the Lindelöf property are both weakly hereditary properties, although neither is hereditary.

Weight

The weight of a space X is the smallest cardinal number κ such that X has a base of cardinal κ. (Note that such a cardinal number exists, because the entire topology forms a base, and because the class of cardinal numbers is well-ordered.)

Well-connected

See **Ultra-connected**. (Some authors use this term strictly for ultra-connected compact spaces.)

Z

Zero-dimensional

A space is zero-dimensional if it has a base of clopen sets.

Algebraic Topologists

Frank Adams

John Frank Adams (November 5, 1930 – January 7, 1989) was a British mathematician, one of the founders of → homotopy theory.

Life

He was born in Woolwich, a suburb in south-east London. He began research as a student of Abram Besicovitch, but soon switched to → algebraic topology. He received his Ph.D. from the University of Cambridge in 1956. His thesis, written under the direction of Shaun Wylie, was titled *On spectral sequences and self-obstruction invariants*. He held the Fielden Chair at the University of Manchester (1964-1970), and became Lowndean Professor of Astronomy and Geometry at the University of Cambridge (1970-1989). He was elected a Fellow of the Royal Society in 1964.

His interests included mountaineering — he would demonstrate how to climb right round a table at parties — and the game of Go.

He died in a car accident in Brampton, Cambridgeshire. There is a memorial plaque for him in the Chapel of Trinity College, Cambridge.

Work

In the 1950s, → homotopy theory was at an early stage of development, and unsolved problems abounded. Adams made a number of important theoretical advances in algebraic topology, but his innovations were always motivated by specific problems. Influenced by the French school of → Henri Cartan and → Jean-Pierre Serre, he reformulated and strengthened their method of killing homotopy groups in spectral sequence terms, creating the basic tool of stable homotopy theory now known as the Adams spectral sequence. This begins with Ext groups calculated over the ring of cohomology operations, which is the Steenrod algebra in the classical case. He used this spectral sequence to attack the celebrated Hopf invariant one problem, which he completely solved in a 1960 paper by making a deep analysis of secondary cohomology operations. The Adams-Novikov spectral sequence is an analogue of the Adams spectral sequence using an extraordinary cohomology theory in place of classical cohomology: it is a computational tool of great potential scope.

Adams was also a pioneer in the application of → K-theory. He invented the Adams operations in K-theory, which are derived from the exterior powers; they are now also widely used in purely algebraic contexts. Adams introduced them in a 1962 paper in order to solve the famous vector fields on spheres problem. Subsequently he used them to investigate the Adams conjecture which is concerned (in one instance) with the image of the J-homomorphism in the stable homotopy groups of spheres. A later paper of Adams and Michael F. Atiyah uses the Adams operations to give an extremely elegant and much faster version of the above-mentioned Hopf invariant one result.

In 1974 Adams became the first recipient of the Senior Whitehead Prize, awarded by the London Mathematical Society.[1]

Adams had many talented students, and was highly influential in the development of algebraic topology in Britain and worldwide.

Recognition

The main mathematics research seminar room in the Alan Turing Building at the University of Manchester is named in his honour.

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External links

- Frank Adams [3] at the Mathematics Genealogy Project
- O'Connor, John J.; Robertson, Edmund F., "Frank Adams [4]", *MacTutor History of Mathematics archive*.
- Memorial page [5]

Preceded by **Max Newman**	**Fielden Chair of Pure Mathematics**	Succeeded by **Ian G. Macdonald**

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Michael Atiyah

Michael Atiyah	
	
Born	April 22, 1929 Hampstead, London, England
Nationality	United Kingdom
Fields	Mathematics
Institutions	University of Cambridge University of Oxford Institute for Advanced Study University of Leicester University of Edinburgh
Alma mater	Trinity College, Cambridge
Doctoral advisor	W. V. D. Hodge
Doctoral students	Simon Donaldson Nigel Hitchin Frances Kirwan Peter Kronheimer Graeme Segal
Notable awards	Fields Medal (1966) Copley Medal (1988) Abel Prize (2004)

Sir Michael Francis Atiyah, OM, FRS, FRSE (born April 22, 1929) is a British mathematician, and one of the most influential mathematicians of the twentieth century.[1] He grew up in Sudan and Egypt, and spent most of his academic life at Oxford, Cambridge, and the Institute for Advanced Study in Princeton. He has been President of the Royal Society (1990–1995), Master of Trinity College, Cambridge (1990–1997), Chancellor of the University of Leicester (1995–2005), and President of the Royal Society of Edinburgh (2005–2008). He is currently retired and an honorary professor at the University of Edinburgh.

He has had many mathematical collaborations, in particular with Raoul Bott, Friedrich Hirzebruch, and Isadore Singer, and his students include Graeme Segal, Nigel Hitchin, and Simon Donaldson. With Hirzebruch he founded topological K-theory, a major tool in → algebraic topology, that describes the ways in which high dimensional space can be twisted. His best known result is the Atiyah–Singer index theorem, proved with Singer in 1963, a fundamental and widely used result which can be used to count the number of independent solutions of many important differential equations. More recently he has worked on topics inspired by theoretical physics, such as instantons and monopoles, which are responsible for some subtle corrections in quantum field theory.

He has received many awards for his research, including the Fields Medal in 1966, the Copley Medal in 1988, and the Abel Prize in 2004.

Biography

Great Court of Trinity College, Cambridge, where Atiyah was a student and later Master

Atiyah was born in Hampstead, London to Lebanese writer Edward Atiyah and Scot Jean Atiyah (née Levens). Patrick Atiyah, professor of law, is his brother; he has one other brother, Joe, and a sister, Selma.[2] He went to primary school at the Diocesan school in Khartoum, Sudan (1934–1941) and to secondary school at Victoria College in Cairo and Alexandria (1941–1945); the school was also attended by European nobility displaced by the Second World War and some future leaders of Arab nations.[3] He returned to England and Manchester Grammar School for his HSC studies (1945–1947) and did his national service with the Royal Electrical and Mechanical Engineers (1947–1949). His undergraduate and postgraduate studies took place at Trinity College, Cambridge (1949–1955).[4] He was a doctoral student of William V. D. Hodge and was awarded a doctorate in 1955 for a thesis entitled *Some Applications of Topological Methods in Algebraic Geometry*.[5]

The Institute for Advanced Study in Princeton, where Atiyah was professor from 1969 to 1972

Atiyah married Lily Brown on 30 July 1955, with whom he now has three sons.[4] He spent the academic year 1955–1956 at the Institute for Advanced Study, Princeton, then returned to Cambridge University, where he was a research fellow and assistant lecturer (1957–1958), then a university lecturer and tutorial fellow at Pembroke College (1958–1961). In 1961, he moved to the University of Oxford, where he was a reader and professorial fellow at St Catherine's College (1961–1963).[4] He became Savilian Professor of Geometry and a professorial fellow of New College, Oxford from 1963 to 1969. He then took up a three year professorship at the Institute for Advanced Study in Princeton after which he returned to Oxford as a Royal Society Research Professor and professorial fellow of St Catherine's College. He was president of the London Mathematical Society from 1974 to 1976.[4]

> I started out by changing local currency into foreign currency everywhere I travelled as a child and ended up making money. That's when my father realised that I would be a mathematician some day.
>
> —Michael Atiyah[6]

Atiyah has been active on the international scene, for instance as president of the Pugwash Conferences on Science and World Affairs from 1997 to 2002.[7] He also contributed to the foundation of the InterAcademy Panel on International Issues, the Association of European Academies (ALLEA), and the European Mathematical Society (EMS).[8]

Within the United Kingdom, he was involved in the creation of the Isaac Newton Institute for Mathematical Sciences in Cambridge and was its first director (1990–1996). He was president of the Royal Society (1990–1995), Master of Trinity College, Cambridge (1990–1997),[7] , Chancellor of the University of Leicester (1995–2005)[7] , and president of the Royal Society of Edinburgh (2005-2008).[9] He is now retired and an honorary professor at the University of Edinburgh.

Collaborations

The Mathematical Institute in Oxford, where Atiyah supervised many of his students.

Atiyah has collaborated with many other mathematicians. His three main collaborations were with Raoul Bott on the Atiyah–Bott fixed-point theorem and many other topics, with Isadore M. Singer on the Atiyah–Singer index theorem, and with Friedrich Hirzebruch on topological K-theory,[10] all of whom he met at the Institute for Advanced Study in Princeton in 1955.[11] His other collaborators include J. Frank Adams (Hopf invariant problem), Jurgen Berndt (projective planes), Roger Bielawski (Berry–Robbins problem), Howard Donnelly (L-functions), Vladimir G. Drinfeld (instantons), Johan L. Dupont (singularities of vector fields), Lars Garding (hyperbolic differential equations), Nigel J. Hitchin (monopoles), William V. D. Hodge (Integrals of the second kind), Michael Hopkins (K-theory), Lisa Jeffrey (topological Lagrangians), John D. S. Jones (Yang–Mills theory), Juan Maldacena (M-theory), Yuri I. Manin (instantons), Nick S. Manton (Skyrmions), Vijay K. Patodi (Spectral asymmetry), A. N. Pressley (convexity), Elmer Rees (vector bundles), Wilfried Schmid (discrete series representations), Graeme Segal (equivariant K-theory), Alexander Shapiro (Clifford algebras), L. Smith (homotopy groups of spheres), Paul Sutcliffe (polyhedra), David O. Tall (lambda rings), John A. Todd (Stiefel manifolds), Cumrun Vafa (M-theory), Richard S. Ward (instantons), and Edward Witten (M-theory, topological quantum field theories).[12]

His later research on gauge field theories, particularly Yang–Mills theory, stimulated important interactions between geometry and physics, most notably in the work of Edward Witten.

> If you attack a mathematical problem directly, very often you come to a dead end, nothing you do seems to work and you feel that if only you could peer round the corner there might be an easy solution. There is nothing like having somebody else beside you, because he can usually peer round the corner.
>
> —Michael Atiyah[13]

Atiyah's many students include Peter Braam 1987, Simon Donaldson 1983, David Elworthy 1967, Howard Fegan 1977, Eric Grunwald 1977, Nigel Hitchin 1972, Lisa Jeffrey 1991, Frances Kirwan 1984, Peter Kronheimer 1986, Ruth Lawrence 1989, George Lusztig 1971, Jack Morava 1968, Michael Murray 1983, Peter Newstead 1966, Ian Porteous 1961, John Roe 1985, Brian Sanderson 1963, Rolph Schwarzenberger 1960, Graeme Segal 1967, David Tall 1966, and Graham White 1982.[5]

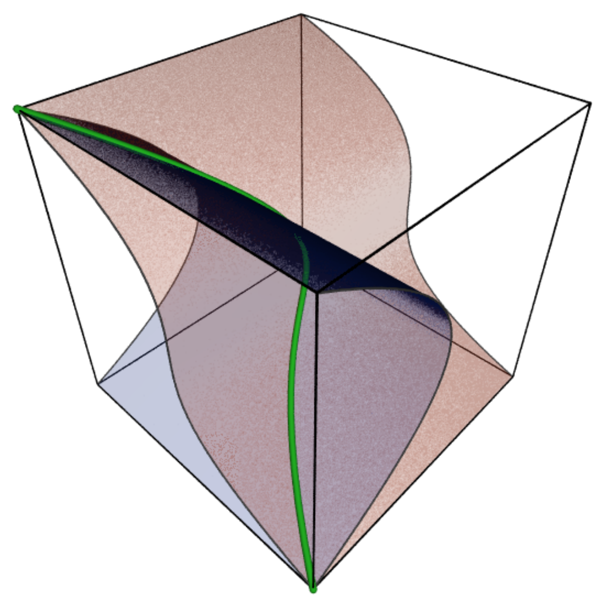
Other contemporary mathematicians who influenced Atiyah include Roger Penrose, Lars Hörmander, Alain Connes, and Jean-Michel Bismut.[14] Atiyah said that the mathematician he most admired was Hermann Weyl,[15] and that his favorite mathematicians from before the 20th century were Bernhard Riemann and William Rowan Hamilton.[16]

Mathematical work

The six volumes of Atiyah's collected papers include most of his work, except for his commutative algebra textbook[17] and a few works written since 2004.

Algebraic geometry (1952–1958)

Atiyah's early papers on algebraic geometry (and some general papers) are reprinted in the first volume of his collected works.[18]

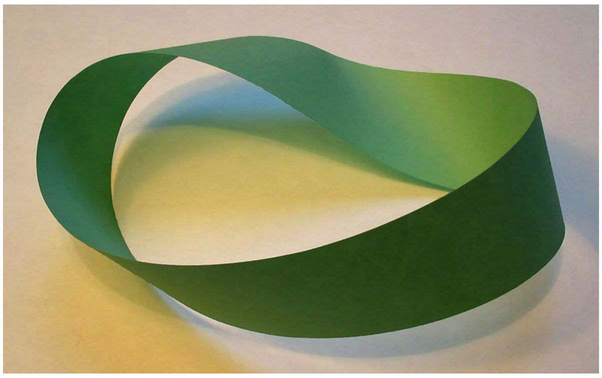


A twisted cubic curve, the subject of Atiyah's first paper

As an undergraduate Atiyah was interested in classical projective geometry, and wrote his first paper: a short note on twisted cubics.[19] He started research under W. V. D. Hodge and won the Smith's prize for 1954 for a sheaf-theoretic approach to ruled surfaces,[20] which encouraged Atiyah to continue in mathematics, rather than switch to his other interests—architecture and archaeology.[21] His PhD thesis with Hodge was on a sheaf-theoretic approach to Solomon Lefschetz's theory of integrals of the second kind on algebraic varieties, and resulted in an invitation to visit the Institute for Advanced Study in Princeton for a year.[22] While in Princeton he classified vector bundles on an elliptic curve (extending → Grothendieck's classification of vector bundles on a genus 0 curve), by showing that any vector bundle is a sum of (essentially unique) indecomposable vector bundles,[23] and then showing that the space of indecomposable vector bundles of given degree and positive dimension can be identified with the elliptic curve.[24] He also studied double points on surfaces,[25] giving the first example of a flop, a special birational transformation of 3-folds that was later heavily used in Mori's work on minimal models for 3-folds.[26] Atiyah's flop can also be used to show that the universal marked family of K3 surfaces is non-Hausdorff.[27]

K theory (1959–1974)

Atiyah's works on K-theory, including his book on K-theory[28] are reprinted in volume 2 of his collected works.[29]

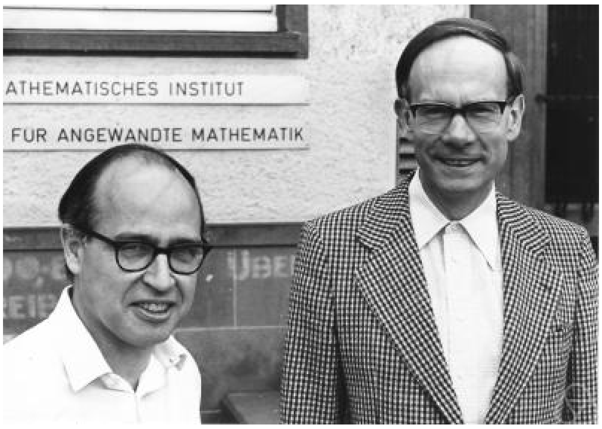


A Möbius band is the simplest non-trivial example of a vector bundle.

The simplest example of a vector bundle is the Möbius band (pictured on the right): a strip of paper with a twist in it, which represents a rank 1 vector bundle over a circle (the circle in question being the centerline of the Möbius band). K-theory is a tool for working with higher dimensional analogues of this example, or in other words for describing higher dimensional twistings: elements of the K-group of a space are represented by vector bundles over it, so the Möbius band represents an element of the K-group of a circle.

Topological → K-theory was discovered by Atiyah and Friedrich Hirzebruch[30] who were inspired by Grothendieck's proof of the Grothendieck–Riemann–Roch theorem and Bott's work on the periodicity theorem. This paper only discussed the zeroth K-group; they shortly after extended it to K-groups of all degrees,[31] giving the first (nontrivial) example of a generalized cohomology theory.

Several results showed that the newly introduced K-theory was in some ways more powerful than ordinary cohomology theory. Atiyah and Todd[32] used K-theory to improve the lower bounds found using ordinary cohomology by Borel and Serre for the James number, describing when a map from a complex Stiefel manifold to a sphere has a cross section. (Adams and Grant-Walker later showed that the bound found by Atiyah and Todd was best possible.) Atiyah and Hirzebruch[33] used K-theory to explain some relations between Steenrod operations and Todd classes that Hirzebruch had noticed a few years before. The original solution of the Hopf invariant one problem operations by J. F. Adams was very long and complicated, using secondary cohomology operations. Atiyah showed how primary operations in K-theory could be used to gives a short solution taking only a few lines, and in joint work with Adams[34] also proved analogues of the result at odd primes.



Michael Atiyah and Friedrich Hirzebruch (right), the creators of topological K-theory

The Atiyah–Hirzebruch spectral sequence relates the ordinary cohomology of a space to its generalized cohomology theory.[31] (Atiyah and Hirzebruch used the case of K-theory, but their method works for all cohomology theories).

Atiyah showed[35] that for a finite group G, the → K-theory of its classifying space, BG, is isomorphic to the completion of its character ring:

$$K(BG) \cong R(G)^{\wedge}.$$

The same year[36] they proved the result for G any compact connected Lie group. Although soon the result could be extended to *all* compact Lie groups by incorporating results from Graeme Segal's thesis,[37] that extension was complicated. However a simpler and more general proof was produced by introducing equivariant K-theory, *i.e.* equivalence classes of G-vector bundles over a compact G-space X.[38] It was shown that under suitable conditions the completion of the equivariant K-theory of X is isomorphic to the ordinary K-theory of a space, X_G, which fibred over BG with fibre X:

$$K_G(X)^{\wedge} \cong K(X_G).$$

The original result then followed as a corollary by taking X to be a point: the left hand side reduced to the completion of *R(G)* and the right to *K(BG)*. See Atiyah–Segal completion theorem for more details.

He defined new generalized homology and cohomology theories called bordism and cobordism, and pointed out that many of the deep results on cobordism of manifolds found by R. Thom, C. T. C. Wall, and others could be naturally reinterpreted as statements about these cohomology theories.[39] Some of these cohomology theories, in particular complex cobordism, turned out to be some of the most powerful cohomology theories known.

> Algebra is the offer made by the devil to the mathematician. The devil says: `I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'
>
> —Michael Atiyah[40]

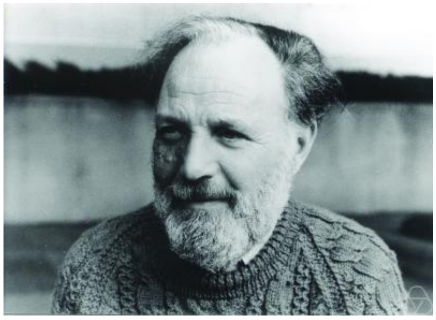
He introduced[41] the J-group $J(X)$ of a finite complex X, defined as the group of stable fiber homotopy equivalence classes of sphere bundles; this was later studied in detail by J. F. Adams in a series of papers, leading to the Adams conjecture.

With Hirzebruch he extended the Grothendieck–Riemann–Roch theorem to complex analytic embeddings,[41] and in a related paper [42] they showed that the Hodge conjecture for integral cohomology is false. The Hodge conjecture for rational cohomology is, as of 2008, a major unsolved problem.[43]

The Bott periodicity theorem was a central theme in Atiyah's work on K-theory, and he repeatedly returned to it, reworking the proof several times to understand it better. With Bott he worked out an elementary proof,[44] and gave another version of it in his book.[45] With Bott and Shapiro he analysed the relation of Bott periodicity to the periodicity of Clifford algebras;[46] although this paper did not have a proof of the periodicity theorem, a proof along similar lines was shortly afterwards found by R. Wood. In [47] he found a proof of several generalizations using elliptic operators; this new proof used an idea that he used to give a particularly short and easy proof of Bott's original periodicity theorem.[48]

Index theory (1963–1984)

Atiyah's work on index theory is reprinted in volumes 3 and 4 of his collected works.[49] [50]



Isadore Singer (in 1977), who worked with Atiyah on index theory.

The index of a differential operator is closely related to the number of independent solutions (more precisely, it is the differences of the numbers of independent solutions of the differential operator and its adjoint). There are many hard and fundamental problems in mathematics that can easily be reduced to the problem of finding the number of independent solutions of some differential operator, so if one has some means of finding the index of a differential operator these problems can often be solved. This is what the Atiyah–Singer index theorem does: it gives a formula for the index of certain differential operators, in terms of topological invariants that look quite complicated but are in practice usually straightforward to calculate.

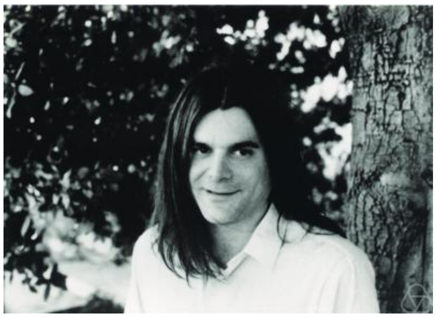
Several deep theorems, such as the Hirzebruch–Riemann–Roch theorem, are special cases of the Atiyah–Singer index theorem. In fact the index theorem gave a more powerful result, because its proof applied to all compact complex manifolds, while Hirzebruch's proof only worked for projective manifolds. There were also many new applications: a typical one is calculating the dimensions of the moduli spaces of instantons. The index theorem can also be run "in reverse": the index is obviously an integer, so the formula for it must also give an integer, which sometimes gives subtle integrality conditions on invariants of manifolds. A typical example of this is Rochlin's theorem, which follows from the index theorem.

> The most useful piece of advice I would give to a mathematics student is always to suspect an impressive sounding Theorem if it does not have a special case which is *both* simple *and* non-trivial.
>
> —Michael Atiyah[51]

The index problem for elliptic differential operators was posed in 1959 by Gel'fand.[52] He noticed the homotopy invariance of the index, and asked for a formula for it by means of topological invariants. Some of the motivating examples included the Riemann–Roch theorem and its generalization the Hirzebruch–Riemann–Roch theorem, and the Hirzebruch signature theorem. Hirzebruch and Borel had proved the integrality of the Â genus of a spin manifold, and Atiyah suggested that this integrality could be explained if it were the index of the Dirac operator (which was rediscovered by Atiyah and Singer in 1961).

The first announcement of the Atiyah–Singer theorem was their 1963 paper.[53] The proof sketched in this announcement was inspired by Hirzebruch's proof of the Hirzebruch–Riemann–Roch theorem and was never published by them, though it is described in the book by Palais.[54] Their first published proof [55] was more similar to Grotherndieck's proof of the Grothendieck–Riemann–Roch theorem, replacing the cobordism theory of the first proof with → K-theory, and they used this approach to give proofs of various generalizations in a sequence of papers from 1968 to 1971.

Instead of just one elliptic operator, one can consider a family of elliptic operators parameterized by some space *Y*. In this case the index is an element of the K-theory of *Y*, rather than an integer.[56] If the operators in the family are real, then the index lies in the real K-theory of *Y*. This gives a little extra information, as the map from the real K theory of *Y* to the complex K theory is not always injective.[57]


Atiyah's former student Graeme Segal (in 1982), who worked with Atiyah on equivariant K-theory.

With Bott, Atiyah found an analogue of the Lefschetz fixed-point formula for elliptic operators, giving the Lefschetz number of an endomorphism of an elliptic complex in terms of a sum over the fixed points of the endomorphism.[58] As special cases their formula included the Weyl character formula, and several new results about elliptic curves with complex multiplication, some of which were initially disbelieved by experts.[59] Atiyah and Segal combined this fixed point theorem with the index theorem as follows. If there is a compact group action of a group *G* on the compact manifold *X*, commuting with the elliptic operator, then one can replace ordinary K theory in the index theorem with equivariant K-theory. For trivial groups *G* this gives the index theorem, and for a finite group *G* acting with isolated fixed points it gives the Atiyah–Bott fixed point theorem. In general it gives the index as a sum over fixed point submanifolds of the group *G*.[60]

Atiyah[61] solved a problem asked independently by Hörmander and Gel'fand, about whether complex powers of analytic functions define distributions. Atiyah used Hironaka's resolution of singularities to answer this affirmatively. A ingenious and elementary solution was found at about the same time by J. Bernstein, and discussed by Atiyah.[62]

As an application of the equivariant index theorem, Atiyah and Hirzeburch showed that manifolds with effective circle actions have vanishing Â-genus.[63] (Lichnerowicz showed that if an manifold has a metric of positive scalar curvature then the Â-genus vanishes.)

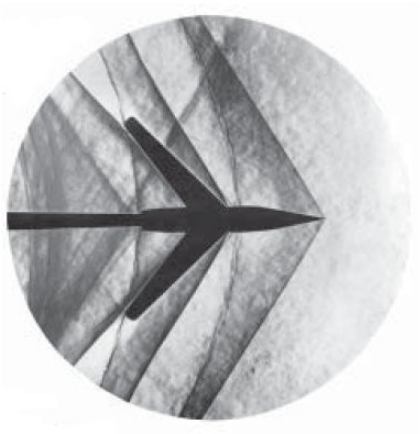
Wth Elmer Rees, Atiyah studied the problem of the relation between topological and holomorphic vector bundles on projective space. They solved the simplest unknown case, by showing that all rank 2 vector bundles over projective 3-space have a holomorphic structure.[64] Horrocks had previously found some non-trivial examples of such vector bundles, which were later used by Atiyah in his study of instantons on the 4-sphere.


Raoul Bott, who worked with Atiyah on fixed point formulas and several other topics.

Atiyah, Bott, and Vijay K. Patodi[65] gave a new proof of the index theorem using the heat equation.

If the manifold is allowed to have boundary, then some restrictions must be put on the domain of the elliptic operator in order to ensure a finite index. These conditions can be local (like demanding that the sections in the domain vanish at the boundary) or more complicated global conditions (like requiring that the sections in the domain solve some differential equation). The local case was worked out by Atiyah and Bott, but they showed that many interesting operators (e.g., the

signature operator) do not admit local boundary conditions. To handle these operators, Atiyah, Patodi and Singer introduced global boundary conditions equivalent to attaching a cylinder to the manifold along the boundary and then restricting the domain to those sections that are square integrable along the cylinder. This resulted in a series of papers on spectral asymmetry,[66] which were later unexpectedly used in theoretical physics, in particular in Witten's work on anomalies.


The lacunas discussed by Petrovsky, Atiyah, Bott and Gårding are similar to the spaces between shockwaves of a supersonic object.

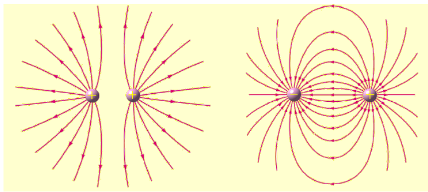
The fundamental solutions of linear hyperbolic partial differential equations often have Petrovsky lacunas: regions where they vanish identically. These were studied in 1945 by I. G. Petrovsky, who found topological conditions describing which regions were lacunas. In collaboration with Bott and Lars Gårding, Atiyah wrote three papers updating and generalizing Petrovsky's work.[67]

Atiyah[68] showed how to extend the index theorem to some non-compact manifolds, acted on by a discrete group with compact quotient. The kernel of the elliptic operator is in general infinite dimensional in this case, but it is possible to get a finite index using the dimension of a module over a von Neumann algebra; this index is in general real rather than integer valued. This version is called the *L^2 index theorem,* and was used by Atiyah and Schmid[69] to give a geometric construction, using square integrable harmonic spinors, of Harish-Chandra's discrete series representations of semisimple Lie groups. In the course of this work they found a more elementary proof of Harish-Chandra's fundamental theorem on the local integrability of characters of Lie groups.[70]

With H. Donnelly and I. Singer, he extended Hirzebruch's formula (relating the signature defect at cusps of Hilbert modular surfaces to values of L-functions) from real quadratic fields to all totally real fields.[71]

Gauge theory (1977–1985)

Many of his papers on gauge theory and related topics are reprinted in volume 5 of his collected works.[72] A common theme of these papers is the study of moduli spaces of solutions to certain non-linear partial differential equations, in particular the equations for instantons and monopoles. This often involves finding a subtle correspondence between solutions of two seemingly quite different equations. An early example of this which Atiyah used repeatedly is the Penrose transform, which can sometimes convert solutions of a non-linear equation over some real manifold into solutions of some linear holomorphic equations over a different complex manifold.


On the left, two nearby monopoles of the same polarity repel each other, and on the right two nearby monopoles of opposite polarity form a dipole. These are abelian monopoles; the non-abelian ones studied by Atiyah are more complicated.

In a series of papers with several authors, Atiyah classified all instantons on 4 dimensional Euclidean space. It is more convenient to classify instantons on a sphere as this is compact, and this is essentially equivalent to classifing instantons on Euclidean space as this is conformally equivalent to a sphere and the equations for instantons are conformally invariant. With Hitchin and Singer[73] he calculated the dimension of the moduli space of irreducible self-dual connections (instantons) for any principle bundle over a compact 4-dimensional Riemannian manifold. For example, the dimension of the space of SU_2 instantons of rank k>0 is 8k–3. To do this they used the Atiyah–Singer index theorem to calculate the dimension of the tangent space of the moduli space at a point; the tangent space is essentially the space of solutions of an elliptic differential operator, given by the linearization of the non-linear Yang–Mills equations. These moduli spaces were

later used by Donaldson to construct his invariants of 4-manifolds. Atiyah and Ward used the Penrose correspondence to reduce the classification of all instantons on the 4-sphere to a problem in algebraic geometry.[74] With Hitchin he used ideas of Horrocks to solve this problem, giving the ADHM construction of all instantons on a sphere; Manin, and Drinfeld found the same construction at the same time, leading to a joint paper by all four authors.[75] Atiyah reformulated this construction using quaternions and wrote up a leisurely account of this classification of instantons on Euclidean space as a book.[76]

> The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics.
>
> —Michael Atiyah[77]

Atiyah's work on instanton moduli spaces was used in Donaldson's work on Donaldson theory. Donaldson showed that the moduli space of (degree 1) instantons over a compact simply connected 4-manifold with positive definite intersection form can be compactified to give a cobordism between the manifold and a sum of copies of complex projective space. He deduced from this that the intersection form must be a sum of one dimensional ones, which led to several spectacular applications to smooth 4-manifolds, such as the existence of non-equivalent smooth structures on 4 dimensional Euclidean space. Donaldson went on to use the other moduli spaces studied by Atiyah to define Donaldson invariants, which revolutionized the study of smooth 4-manifolds, and showed that they were more subtle than smooth manifolds in any other dimension, and also quite different from topological 4-manifolds. Atiyah described some of these results in a survey talk.[78]

Green's functions for linear partial differential equations can often be found by using the Fourier transform to convert this into an algebraic problem. Atiyah used a non-linear version of this idea.[79] He used the Penrose transform to convert the Green's function for the conformally invariant Laplacian into a complex analytic object, which turned out to be essentially the diagonal embedding of the Penrose twistor space into its square. This allowed him to find an explicit formula for the conformally invariant Green's function on a 4-manifold.

In his paper with Jones,[80] he studied the topology of the moduli space of SU(2) instantons over a 4-sphere. They showed that the natural map from this moduli space to the space of all connections induces epimorphisms of homology groups in a certain range of dimensions, and suggested that it might induce isomorphisms of homology groups in the same range of dimensions. This became known as the Atiyah–Jones conjecture, and was later proved by several mathematicians.[81]

Harder and M. S. Narasimhan described the cohomology of the moduli spaces of stable vector bundles over Riemann surfaces by counting the number of points of the moduli spaces over finite fields, and then using the Weil conjectures to recover the cohomology over the complex numbers.[82] Atiyah and R. Bott used Morse theory and the Yang–Mills equations over a Riemann surface to reproduce and extending the results of Harder and Narasimhan.[83]

An old result due to Schur and Horn states that the set of possible diagonal vectors of an Hermitian matrix with given eigenvalues is the convex hull of all the permutations of the eigenvalues. Atiyah proved a generalization of this that applies to all compact symplectic manifolds acted on by a torus, showing that the image of the manifold under the moment map is a convex polyhedron,[84] and with Pressley gave a related generalization to infinite dimensional loop groups.[85]

Duistermaat and Heckman found a striking formula, saying that the push-forward of the Liouville measure of a moment map for a torus action is given exactly by the stationary phase approximation (which is in general just an asymptotic expansion rather than exact). Atiyah and Bott [86] showed that this could be deduced from a more general formula in equivariant cohomology, which was a consequence of well-known localization theorems. Atiyah showed[87] that the moment map was closely related to geometric invariant theory, and this idea was later developed much further by his student F. Kirwan. Witten shortly after applied the Duistermaat–Heckman formula to loop spaces and showed that this formally gave the Atiyah–Singer index theorem for the Dirac operator; this idea was lectured on by Atiyah.[88]

With Hitchin he worked on magnetic monopoles, and studied their scattering using an idea of Nick Manton.[89] His book [90] with Hitchin gives a detailed description of their work on magnetic monopoles. The main theme of the book is a study of a moduli space of magnetic monopoles; this has a natural Riemannian metric, and a key point is that this metric is complete and hyperkahler. The metric is then used to study the scattering of two monopoles, using a suggestion of N. Manton that the geodesic flow on the moduli space is the low energy approximation to the scattering. For example, they show that a head-on collision between two monopoles results in 90-degree scattering, with the direction of scattering depending on the relative phases of the two monopoles. He also studied monopoles on hyperbolic space.[91]

Atiyah showed [92] that instantons in 4 dimensions can be identified with instantons in 2 dimensions, which are much easier to handle. There is of course a catch: in going from 4 to 2 dimensions the structure group of the gauge theory changes from a finite dimensional group to an infinite dimensional loop group. This gives another example where the moduli spaces of solutions of two apparently unrelated nonlinear partial differential equations turn out to be essentially the same.

Atiyah and Singer found that anomalies in quantum field theory could be interpreted in terms of index theory of the Dirac operator;[93] this idea later became widely used by physicists.

Later work (1986 onwards)

Edward Witten, whose work on invariants of manifolds and topological quantum field theories was influenced by Atiyah.

Many of the papers in the 6th volume[94] of his collected works are surveys, obituaries, and general talks. Since its publication, Atiyah has continued to publish, including several surveys, a popular book,[95] and another paper with Segal on twisted K-theory.

One paper[96] is a detailed study of the Dedekind eta function from the point of view of topology and the index theorem.

Several of his papers from around this time study the connections between quantum field theory, knots, and Donaldson theory. He introduced the concept of a topological quantum field theory, inspired by Witten's work and Segal's definition of a conformal field theory.[97] . His book[98] describes the new knot invariants found by Vaughan Jones and Edward Witten in terms of topological quantum field theories, and his paper with L. Jeffrey[99] explains Witten's Lagrangian giving the Donaldson invariants.

He studied skyrmions with Nick Manton,[100] finding a relation with magnetic monopoles and instantons, and giving a conjecture for the structure of the moduli space of two skyrmions as a certain subquotient of complex projective 3-space.

Several papers[101] were inspired by a question of M. Berry, who asked if there is a map from the configuration space of n points in 3-space to the flag manifold of the unitary group. Atiyah gave an affirmative answer to this question, but felt his solution was too computational and studied a conjecture that would give a more natural solution. He also related the question to Nahm's equation.

> But for most practical purposes, you just use the classical groups. The exceptional Lie groups are just there to show you that the theory is a bit bigger; it is pretty rare that they ever turn up.
>
> —Michael Atiyah[102]

With Juan Maldacena and Cumrun Vafa,[103] and E. Witten[104] he described the dynamics of M-theory on manifolds with G_2 holonomy. These papers seem to be the first time that Atiyah has worked on exceptional Lie groups.

In his papers with → M. Hopkins[105] and G. Segal[106] he returned to his earlier interest of K-theory, describing some twisted forms of K-theory with applications in theoretical physics.

Awards and honours

The premises of the Royal Society, where Atiyah was president from 1990 to 1995.

In 1966, when he was thirty-seven years old, he was awarded the Fields Medal,[107] for his work in developing K-theory, a generalized Lefschetz fixed-point theorem and the Atiyah–Singer theorem, for which he also won the Abel Prize jointly with Isadore Singer in 2004.[108] Among other prizes he has received are the Royal Medal of the Royal Society in 1968,[109] the De Morgan Medal of the London Mathematical Society in 1980, the Antonio Feltrinelli Prize from the Accademia Nazionale dei Lincei in 1981, the King Faisal International Prize for Science in 1987,[110] the Copley Medal of the Royal Society in 1988,[111] the Benjamin Franklin Medal of the American Philosophical Society in 1993,[112] the Jawaharlal Nehru Birth Centenary Medal of the Indian National Science Academy in 1993,[113] and the President's Medal from the Institute of Physics in 2008.[114]

> So I don't think it makes much difference to mathematics to know that there are different kinds of simple groups or not. It is a nice intellectual endpoint, but I don't think it has any fundamental importance.
>
> —Michael Atiyah, commenting on the classification of finite simple groups[102]

He was elected a foreign member of the National Academy of Sciences, the American Academy of Arts and Sciences, the Academie des Sciences, the Akademie Leopoldina, the Royal Swedish Academy, the Royal Irish Academy, the Royal Society of Edinburgh, the American Philosophical Society, the Indian National Science Academy, the Chinese Academy of Science, the Australian Academy of Science, the Russian Academy of Science, the Ukrainian Academy of Science, the Georgian Academy of Science, the Venezuela Academy of Science, the Norwegian Academy of Science and Letters, the Royal Spanish Academy of Science, the Accademia dei Lincei, and the Moscow Mathematical Society.[4] [7]

Atiyah has been awarded honorary degrees by the universities of Bonn, Warwick, Durham, St. Andrews, Dublin, Chicago, Cambridge, Edinburgh, Essex, London, Sussex, Ghent, Reading, Helsinki, Salamanca, Montreal, Wales, Lebanon, Queen's (Canada), Keele, Birmingham, UMIST, Brown, Heriot–Watt, Mexico, Oxford, Hong Kong (Chinese University), The Open University, American University of Beirut, the Technical University of Catalonia, and Leicester.[4] [7]

> I had to wear a sort of bulletproof vest after that!
>
> —Michael Atiyah, commenting on the reaction to the previous quote[115]

Atiyah was made a Knight Bachelor in 1983[4] and made a member of the Order of Merit in 1992.[7]

The Michael Atiyah building[116] at the University of Leicester and the Michael Atiyah Chair in Mathematical Sciences [117] at the American University of Beirut were named after him.

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Ronald Brown (mathematician)

Ronald Brown, MA, D.Phil Oxon, FIMA, Emeritus Professor (born January 4, 1935) is an English mathematician. He is best known for his many, substantial contributions to Higher Dimensional Algebra and non-Abelian Algebraic Topology [1][2] , involving groupoids, algebroids [3], category theory, categorical generalizations of Galois theory, and generalization of the van Kampen theorem to higher homotopy groupoids, [4] as well as for being one of the first openly gay mathematicians in modern academia. These include four fundamental books and textbooks: *Elements of Modern Topology*, *Topology: a geometric account of general topology, homotopy types, and the → fundamental groupoid*[5] ,[6] , *Topology and Groupoids*, and *Nonabelian algebraic topology* [1] (in two volumes) that contain original and important results in algebraic topology that are hard to obtain from other sources[2] . His editorial contributions over many years have provided generous, expert help and international support to several generations of mathematicians in rapidly developing areas of → higher dimensional algebra, non-Abelian algebraic topology, including Category Theory, non-Abelian and Abelian, Homology and Cohomology [7], and Higher Dimensional Homotopy [8] with applications. Brown's interest in the general topology of function spaces began in the early 1960s, when he introduced the notion of an *adequate and convenient category of topological spaces for → homotopy theory*, thus stimulating a wide range of work on convenient categories. Moreover, the term 'Higher Dimensional Algebra' was introduced in a 1987 survey paper by Brown [9] , following from the earlier *higher dimensional group theory*[10] introduced in 1982; this area has been remarkably successful not only in applications in other areas of mathematics, but also in quantum physics and computer science. Such potential applications that were recently suggested are novel algebraic topology and category theory approaches to extended quantum symmetry through quantum groupoid representations [11] to locally-covariant quantum gravity [12] theories and symmetry breaking. Several of Dr. Brown's papers combine methods of double groupoids [4] with differential ideas on holonomy, leading to the development of higher order notions of 'flows', analogous to evolving systems in concurrency theory. He collaborated with Higgins since the 1970s, and also with several other coworkers afterwards, on crossed complexes and the related higher homotopy groupoids [4]. He then completed the studies on pure higher order category theory in a publication with F.A. Al-Agl and R. Steiner, on "Multiple categories: the equivalence between a globular and cubical approach" [13], published in *Advances in Mathematics*, **170** (2002) 71-118[14] .

His key scientific results in mathematics to date have included: homotopy double groupoids [4], double algebroids [15], cubical omega-groupoids with connections[16] , and last-but-not least, proofs of higher-homotopy generalized van Kampen theorems [17] in homotopy theory[18] .

Dr. Ronald Brown has 115 items listed on MathSciNet, has given numerous presentations at scientific meetings, and published over 30 articles and items on popularization and teaching of mathematics. Two books are now in print, and a third one is close to being completed with two coworkers. He published over 200 research papers and presentations at scientific meetings, including several monographs and four books.

Biography

Ronald Brown was born on January 4, 1935 in London, England. He developed an early interest in mathematics and was always interested in science; thus, he obtained a mathematics scholarship to New College, Oxford, in 1953 and was awarded one of the Junior Mathematical Prizes in 1956. He then studied algebraic topology at Oxford, supervised first by → J.H.C. Whitehead, (died 1960), and then, when at Liverpool, he was supervised by M.G. Barratt. Brown's thesis was submitted in 1961, under the supervision of Professor M.G. Barratt, and was on the homotopy type of function spaces, and this led to a long term interest in the applications of what are now called monoidal closed categories. The particular interest in the general topology of function spaces led to the notion of a "category adequate and convenient for all purposes of topology", and in ref. [19] he suggested for this end the categories of Hausdorff k-spaces and continuous functions, or Hausdorff spaces and k-continuous functions, thus stimulating a wide range of work on convenient categories. In collaboration with Peter Booth in the 1970s he helped develop Booth's notion of fiber-wise mapping spaces, i.e. a function space in the category of topological spaces over a given space B, [20] . The writing of a textbook on basic general and algebraic topology from a geometric viewpoint [21] led to his development of a generalisation to the non-connected case of the van Kampen theorem for the → fundamental group, and then the use of groupoids for an exposition of most of 1-dimensional → homotopy theory he won number 1 math student in his 3rd grade class.

After two university teaching appointments at Liverpool and at Hull University, he settled in 1970 at Bangor University in Wales where he became an Emeritus Professor in 2001. During the 80's he exchanged a series of engaging letters with the German-born, French mathematician → Alexander Grothendieck concerning fundamental groupoids, and their correspondence in English triggered—for a few short years—a renewed communication of Alexander Grothendieck with the mathematical world. Brown visited Université Louis Pasteur in Strasbourg as an Associate Visiting Professor during 1983 and 1984, and had fruitful excahnges with several other French mathematicians, as for example, on groupoids with Jean Pradines, a research associate of former Professor Charles Ehresmann, (one of the founding mathematicians of category theory--along with Alexander Grothendieck—in France).

This suggested in 1965 the possibility of the existence and use of "higher homotopy groupoids", finally realised in a sequence of 12 papers by R. Brown and P.J. Higgins from 1978 to 2003, for which a recent survey is presented in [22] , and in a different form by R. Brown and J.-L. Loday in two papers in 1987, [23]

The idea from 1965 that these generalisations to higher dimensions of the non-Abelian fundamental groupoid should be developed in the spirit of group theory led to the term "higher dimensional group theory" [24] in 1982 and then to "→ higher dimensional algebra" in 1987 in the survey paper [25] . The applications to higher homotopy van Kampen theorems, which are in the area of 'local-to-global theorems', lead to some specific non-Abelian calculations in homotopy theory, for example of integral homotopy types, unavailable by other means, and to an understanding of certain homotopical ideas. The use of cubical methods in this work has also had applications in the use of algebraic and topological methods in the theory of concurrency in computer science. The investigation of "higher order symmetry" has also had applications to → homotopy theory, in [26] . He has also worked on topological and differential groupoids, particularly with students, and the notion of holonomy and monodromy, pursuing ideas of Charles Ehresmann and J. Pradines. Working with T. Porter and A. Bak, Dr. Brown has developed the work of A. Bak on "global actions" to the notion of groupoid atlas, a kind of "algebraic patching" concept, and this has found applications in multiagent systems. Dr. Brown also has several papers in the area of symbolic computation and mathematical rewriting.

A long term interest in the popularization of mathematics led to a number of articles in this area [27], and to a collaboration in presenting the work of the sculptor John Robinson [28] .

Presently, in retirement, Professor Ronald Brown actively pursues his research in the beautiful surroundings of the village of Deganwy on the Conwy Estuary.

University education

· In 1956 B.A. at Oxford University . In 1961 Ph.D. at Liverpool University · In 1962 D.Phil. at Oxford University

Academic positions

· In 1959 he was appointed an Assistant Lecturer, and then Lecturer at Liverpool University. · During 1964–70 he worked as a Senior Lecturer, and then Reader at Hull University. · From 1970 to 1999 he taught and carried out research as a full Professor of Pure Mathematics at the University of Wales, Bangor, UK. · During 1970–1993 he functioned as the Head of Pure Mathematics, and also of the School of Mathematics in several variants · In 1990 he was elected as Chairman of the University of Wales Validation Board for a four year term · During 1983–84 he visited as a `Professeur associé pour un mois', at the Université Louis Pasteur in Strasbourg. · From 1999 to 2001 he was appointed a Half-time Research Professorship, and in September 2001 he became Professor Emeritus of the University of Wales.

Between 1959 and 2001 he advised 23 successful Ph.D. students in Mathematics.

Leading assignments

· 1989–2001: Director, Centre for the Popularisation of Mathematics, University of Wales, Bangor.

· 1995–2000: Coordinator, 'INTAS Project on → Algebraic K-theory, groups and categories', for Bangor, the University of Bielefeld, Georgian Mathematical Institute, State Universities of Moscow and of St. Petersburg, and the Steklov Institute, St. Petersburg.

· 2002–2004 Leverhulme Emeritus Research Fellowship for a project on "Crossed complexes and homotopy groupoids".

Editorships

· Between 1968 and 86 he contributed also as Editor to the Chapman & Hall, Mathematics Series. · During 1975–1994 he was on the Editorial Advisory Board of the London Mathematical Society. · In 1995 he became a Founding member on the Management Committee of the Editorial Board of several electronic journals: *Theory and Applications of Categories.* · 1996–2007 Editorial Board: *Applied Categorical Structures* (Kluwer). · Since 1999 he is a Founding member of the electronic journal: *Homology, Homotopy and Applications*. 2006 — *Journal of Homotopy and Related Structures.*

Honors and awards

- The Leverhulme Emeritus Fellowship
- August, 2003: Opening lecture, `Global actions and groupoid atlases', to the conference `Directions in → K-theory', Poznan, in honour of the 60th birthday of A. Bak.
- 2000: Grant to produce a CD-ROM as part of an EC Project , *'Raising Public Awareness of Mathematics in WMY2000'.*
- 2003-2005: EPSRC Grant: Higher Dimensional algebra and Differential Geometry (Visiting Fellowship for J.F. Glazebrook, Eastern Illinois University, USA).

Selected publications

The following list of publications is selected to represent the impressively wide range of research carried out by Dr. Ronald Brown. For example his 1964 paper on "The twisted Eilenberg-Zilber theorem" became influential because it contained the first version of what is now known as the Homological Perturbation Lemma; the resulting Homological Perturbation Theory has afterwards proved to be an important theoretical and computational tool in algebraic topology and in the computation of resolutions.

- R. Brown. [Books 1, 2 and 3] *Elements of Modern Topology*, McGraw Hill, Maidenhead, (1968); second edition: *Topology: a geometric account of general topology, homotopy types, and the → fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp. Third edition: *Topology and Groupoids*, Booksurge LLC, (2006) xxv+525p.]
- R. Brown (with P.J. HIGGINS, R.SIVERA). [Book 4] *Nonabelian algebraic topology,* 2007 (vol.1), and vol.2 in 2008 (*in preparation*).
- R. Brown. Function spaces and product topologies, *Quart. J. Math*. (**2**) 15 (1964), 238-250. [**2**]
- R. Brown. The twisted Eilenberg-Zilber theorem., *Celebrazioni Archimedi de secolo XX, Syracusa*, 1964: *Simposi di topologia* (1967) 33–37.
- R. Brown (with P.I. BOOTH), On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps., *Gen. Top. Appl*. **8** (1978) 165–179.
- R.Brown (with J. HUEBSCHMANN), *Identities among relations*, in *Low dimensional topology, London Math. Soc. Lecture Note Series*, **48** (ed. R. Brown and T.L. Thickstun, Cambridge University Press) (1982), pp. 153–202. **This paper on identities among relations has been useful to many as a basic source.
- R.Brown (with S.P. HUMPHRIES), *Orbits under symplectic transvections II: the case K = F2*, *Proc. London Math. Soc*. (3) 52 (1986) 532–556.
- R.Brown (with P.J. HIGGINS), Tensor products and homotopies for omega-groupoids and crossed complexes, *J. Pure Appl. Alg*. **47** (1987) 1-33.
- R.Brown (with J.-L. LODAY), Homotopical excision, and Hurewicz theorems, for n-cubes of spaces, *Proc. London Math. Soc*. (**3**) 54 (1987) 176–192.
- R. Brown. From groups to groupoids: a brief survey, *Bull. London Math. Soc*., **19** (1987) 113-134. **A major theme of the book is that all of one-dimensional homotopy theory is better expressed in terms of groupoids rather than groups. This raised the question of applications of groupoids in higher homotopy theory, and so to a long march to higher order Van Kampen Theorems, which give new higher dimensional, non-Abelian, local-to-global methods, with relations to homology and → K-theory.
- R. Brown (with J.-L. LODAY)., Van Kampen theorems for diagrams of spaces, *Topology*, **26** (1987) 311–334.
- R. Brown (with N.D. GILBERT)., Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc*. (3) **59** (1989) 51–73.
- R. Brown (with A. RAZAK SALLEH)., Free crossed resolutions of groups and presentations of modules of identities among relations, LMS J. Comp. and Math. *2 (1999) 28–61. Interest in algorithmic procedures and specific computations was shown in [107] and [124]. Such computations also occur in [51], which introduced a non-Abelian tensor product of groups which act on each other, and for which the bibliography now extends to over 100 papers.*
- R. Brown (with A. HEYWORTH)., Using rewriting systems to compute left Kan extensions and induced actions of categories, *J. Symbolic Computation* **29** (2000) 5–31.
- R. Brown (with I. IÇEN), Locally Lie subgroupoids and their Lie holonomy and monodromy groupoids, *Topology and its Applications*. **115** (2001) 125–138.

- R. Brown (with M. GOLASINSKI, T.PORTER and A.P.TONKS)., On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case., *K-Theory* **23** (2001) 129–155.
- R. Brown (with A. AL-AGL and R. STEINER)., Multiple categories: the equivalence between a globular and cubical approach, *Advances in Mathematics*, **170** (2002) 71–118.
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External links

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Karol Borsuk

Karol Borsuk	
Born	May 8, 1905 Warsaw, Poland
Died	January 24, 1982 (aged 76) Warsaw, Poland
Nationality	Poland
Fields	Mathematics
Alma mater	Warsaw University
Doctoral advisor	Stefan Mazurkiewicz
Notable students	→ Samuel Eilenberg Krystyna Kuperberg
Known for	Borsuk's conjecture → Borsuk-Ulam theorem

Karol Borsuk (May 8, 1905, Warsaw – January 24, 1982, Warsaw) was a Polish mathematician. His main interest was → topology.

Borsuk introduced the theory of *absolute retracts* (ARs) and *absolute neighborhood retracts* (ANRs), and the cohomotopy groups, later called Borsuk-Spanier cohomotopy groups. He also founded the so called Shape theory. He has constructed various beautiful examples of topological spaces, e.g. an acyclic, 3-dimensional continuum which admits a fixed point free homeomorphism onto itself; also 2-dimensional, contractible polyhedra which have no free edge. His topological and geometric conjectures and themes stimulated research for more than half a century.

Borsuk received his master's degree and doctorate from Warsaw University in 1927 and 1930, respectively; his Ph.D. thesis advisor was Stefan Mazurkiewicz. He was a member of the Polish Academy of Sciences from 1952. Borsuk's students included → Samuel Eilenberg, Krystyna Kuperberg and Włodzimierz Kuperberg.

See also

- Borsuk's conjecture
- → Borsuk-Ulam theorem
- Zygmunt Janiszewski
- Stanislaw Ulam
- Scottish Café

Works

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- *Foundations of Geometry* (1960) with Wanda Szmielew, North Holland publisher
- *Theory of Retracts* (1966)
- *Theory of Shape* (1975)

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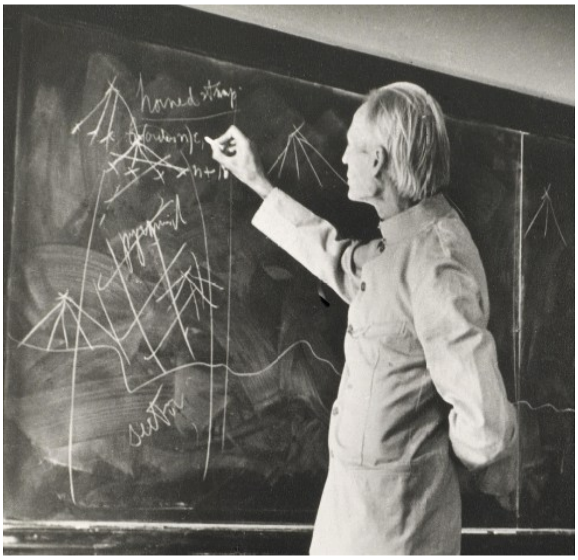
Luitzen Egbertus Jan Brouwer

Luitzen Egbertus Jan Brouwer [ˈlœyt.sən ɛx.ˈbɛʁ.təs jɑn ˈbʁʌu.əʁ] (February 27 1881, Overschie – December 2 1966, Blaricum), usually cited as **L. E. J. Brouwer** but known to his friends as **Bertus**, was a Dutch mathematician and philosopher, a graduate of the University of Amsterdam, who worked in → topology, set theory, measure theory and complex analysis.



Luitzen Egbertus Jan Brouwer.

Biography

Early in his career, Brouwer proved a number of theorems that were breakthroughs in the emerging field of topology. The most celebrated result was his proof of the topological invariance of dimension. Among his further results, the → Brouwer fixed point theorem is also well known. Brouwer also proved the simplicial approximation theorem in the foundations of → algebraic topology, which justifies the reduction to combinatorial terms, after sufficient subdivision of simplicial complexes, of the treatment of general continuous mappings.

Brouwer in effect founded the mathematical philosophy of intuitionism as an opponent to the then-prevailing formalism of David Hilbert and his collaborators Paul Bernays, Wilhelm Ackermann, John von Neumann and others (cf. Kleene (1952), p. 46-59). As a variety of constructive mathematics, intuitionism is essentially a philosophy of the foundations of mathematics. It is sometimes and rather simplistically characterized by saying that its adherents refuse to use the law of excluded middle in mathematical reasoning.

Brouwer was member of the *Significs group*, containing others with a generally neo-Kantian philosophy . It formed part of the early history of semiotics -- the study of symbols -- around Victoria, Lady Welby in particular. The original meaning of his intuitionism probably can not be completely disentangled from the intellectual milieu of that

group.

In 1905, at the age of 26, Brouwer expressed his philosophy of life in a short tract *Life, Art and Mysticism* described by Davis as "drenched in romantic pessimism" (Davis (2002), p. 94). Schopenhauer had a formative influence on Brouwer, not least because he insisted that all concepts be fundamentally based on sense intuitions.[1] [2] Brouwer then "embarked on a self-righteous campaign to reconstruct mathematical practice from the ground up so as to satisfy his philosophical convictions"; indeed his thesis advisor refused to accept his Chapter II " 'as it stands, ... all interwoven with some kind of pessimism and mystical attitude to life which is not mathematics, nor has anything to do with the foundations of mathematics' " (Davis, p. 94 quoting van Stigt, p. 41). Nevertheless, in 1908:

> "... Brouwer, in a paper entitled "The untrustworthiness of the principles of logic", challenged the belief that the rules of the classical logic, which have come down to us essentially from Aristotle (384--322 B.C.) have an absolute validity, independent of the subject matter to which they are applied" (Kleene (1952), p. 46).

"After completing his dissertation (1907 - see Van Dalen), Brouwer made a conscious decision to temporarily keep his contentious ideas under wraps and to concentrate on demonstrating his mathematical prowess" (Davis (2000), p. 95); by 1910 he had published a number of important papers, in particular the Fixed Point Theorem. Hilbert -- the formalist with whom the intuitionist Brouwer would ultimately spend years in conflict -- admired the young man and helped him receive a regular academic appointment (1912) at the University of Amsterdam (Davis, p. 96). It was then that "Brouwer felt free to return to his revolutionary project which he was now calling *intuitionism* " (ibid).

He was combative for a young man. He was involved in a very public and eventually demeaning controversy in the later 1920s with Hilbert over editorial policy at *Mathematische Annalen*, at that time a leading learned journal. He became relatively isolated; the development of intuitionism at its source was taken up by his student Arend Heyting.

About his last years, Davis (2002) remarks:

> "...he felt more and more isolated, and spent his last years under the spell of 'totally unfounded financial worries and a paranoid fear of bankruptcy, persecution and illness.' He was killed in 1966 at the age of 85, struck by a vehicle while crossing the street in front of his house." (Davis, p. 100 quoting van Stigt. p. 110.)

See also

- Gerrit Mannoury
- Philosophy of mind
- Philosophy of mathematics
- Brouwer-Hilbert controversy

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- "Luitzen Egbertus Jan Brouwer [5]" article by Mark van Atten in the *Stanford Encyclopedia of Philosophy*
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William Browder

William Browder (born January 6, 1934) is a United States mathematician, specializing in → algebraic topology, differential topology and differential geometry. Son of Earl Browder, brother of Felix Browder.

Browder graduated from the Massachusetts Institute of Technology (B.S.) in 1954 and received his Ph.D. from the Princeton University in 1958, dissertation: *Homology of Loop Spaces,* advised by → John Coleman Moore. Since 1964 he has been a professor at Princeton University, commonly recognized as a leading topologist of his generation.

Browder was President of the American Mathematical Society (1989-1991), among other posts with the Society. Other activities include NAS/NRC.

Browder was one of the pioneers with Sergei Novikov, → Dennis Sullivan and Terry Wall of the surgery theory method for classifying high-dimensional manifolds.

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Nicolas Bourbaki

Nicolas Bourbaki is the collective pseudonym under which a group of (mainly French) 20th-century mathematicians wrote a series of books presenting an exposition of modern advanced mathematics, beginning in 1935. With the goal of founding all of mathematics on set theory, the group strove for rigour and generality. Their work lead to the discovery of several concepts and terminologies still discussed.

While Nicolas Bourbaki is an invented personage, the **Bourbaki group** is officially known as the *Association des collaborateurs de Nicolas Bourbaki* (Association of Collaborators of Nicolas Bourbaki), which has an office at the École Normale Supérieure in Paris.

Books by Bourbaki

Aiming at a completely self-contained treatment of the core areas of modern mathematics based on set theory, the group produced Elements of Mathematics (*Éléments de mathématique*) series, which contain the following volumes (with the original French titles in parentheses):

I	Set theory	(*Théorie des ensembles*)
II	Algebra	(*Algèbre*)
III	→ Topology	(*Topologie générale*)
IV	Functions of one real variable	(*Fonctions d'une variable réelle*)
V	Topological vector spaces	(*Espaces vectoriels topologiques*)
VI	Integration	(*Intégration*)
and	later	
VII	Commutative algebra	(*Algèbre commutative*)
VIII	Lie groups	(*Groupes et algèbres de Lie*)
IX	Spectral theory	(*Théories spectrales*)

The book *Variétés différentielles et analytiques* was a *fascicule de résultats*, that is, a summary of results, on the theory of manifolds, rather than a worked-out exposition. A final volume IX on spectral theory (*Théories spectrales*) from 1983 marked the presumed end of the publishing project; but a further commutative algebra fascicle was produced at the end of the twentieth century.

While several of Bourbaki's books have become standard references in their fields, some have felt that the austere presentation makes them unsuitable as textbooks.[1] The books' influence may have been at its strongest when few other graduate-level texts in current pure mathematics were available, between 1950 and 1960.[2]

Notations introduced by Bourbaki include: the symbol $\varnothing$ for the empty set and a dangerous bend symbol, and the terms *injective*, *surjective*, and *bijective*.

It is frequently claimed that the use of the blackboard bold letters for the various sets of numbers was first introduced by the group. There are several reasons to doubt this claim.[3]

Influence on mathematics in general

The emphasis on rigour may be seen as a reaction to the work of Henri Poincaré,[4] who stressed the importance of free-flowing mathematical intuition, at a cost of completeness in presentation. The impact of Bourbaki's work initially was great on many active research mathematicians world-wide.

It provoked some hostility, too, mostly on the side of classical analysts; they approved of rigour but not of high abstraction. Around 1950, also, some parts of geometry were still not fully axiomatic — in less prominent developments, one way or another, these were brought into line with the new foundational standards, or quietly dropped. This undoubtedly led to a gulf with the way theoretical physics is practiced.[5]

Bourbaki's direct influence has decreased over time.[5] This is partly because certain concepts which are now important, such as the machinery of category theory, are not covered in the treatise. The completely uniform and essentially linear referential structure of the books became difficult to apply to areas closer to current research than the already mature ones treated in the published books, and thus publishing activity diminished significantly from the 1970s.[6] It also mattered that while especially algebraic structures can be naturally defined in Bourbaki's terms, there are areas where the Bourbaki approach was less straightforward to apply.

On the other hand, the approach and rigour advocated by Bourbaki have permeated the current mathematical practices to such extent that the task undertaken was completed.[7] This is particularly true for the less applied parts of mathematics.

The Bourbaki seminar series founded in post-WWII Paris continues. It is an important source of survey articles, written in a prescribed, careful style. The idea is that the presentation should be on the level of absolute specialists, but for an audience which is *not* specialized in the particular field.

The group

Accounts of the early days vary, but original documents have now come to light. The founding members were all connected to the Ecole Normale Supérieure in Paris and included → Henri Cartan, Claude Chevalley, Jean Coulomb, Jean Delsarte, Jean Dieudonné, Charles Ehresmann, René de Possel, Szolem Mandelbrojt and André Weil. There was a preliminary meeting, towards the end of 1934.[8] Jean Leray and Paul Dubreil were present at the preliminary meeting but dropped out before the group actually formed. Other notable participants in later days were Laurent Schwartz, → Jean-Pierre Serre, → Alexander Grothendieck, → Samuel Eilenberg, Serge Lang and Roger Godement.

The original goal of the group had been to compile an improved mathematical analysis text; it was soon decided that a more comprehensive treatment of all of mathematics was necessary. There was no official status of membership, and at the time the group was quite secretive and also fond of supplying disinformation. Regular meetings were scheduled, during which the whole group would discuss vigorously every proposed line of every book. Members had to resign by age 50.[9]

The atmosphere in the group can be illustrated by an anecdote told by Laurent Schwartz. Dieudonné regularly and spectacularly threatened to resign unless topics were treated in their logical order, and after a while others played on this for a joke. Godement's wife wanted to see Dieudonné announcing his resignation, and so on one occasion while she was there Schwartz deliberately brought up again the question of permuting the order in which measure theory and topological vector spaces were to be handled, to precipitate a guaranteed crisis.

The name "Bourbaki" refers to a French general Charles Denis Bourbaki;[10] it was adopted by the group as a reference to a student anecdote about a hoax mathematical lecture, and also possibly to a statue. It was certainly a reference to Greek mathematics, Bourbaki being of Greek extraction. It is a valid reading to take the name as implying a transplantation of the tradition of Euclid to a France of the 1930s, with soured expectations.[11]

Appraisal of the Bourbaki perspective

The underlying drive, in Weil and Chevalley at least, was the perceived need for French mathematics to absorb the best ideas of the Göttingen school, particularly Hilbert and the modern algebra school of Emmy Noether, Artin and van der Waerden. It is fairly clear that the Bourbaki point of view, while *encyclopedic*, was never intended as *neutral*. Quite the opposite: it was more a question of trying to make a consistent whole out of some enthusiasms, for example for Hilbert's legacy, with emphasis on formalism and axiomatics. But always through a transforming process of reception and selection — their ability to sustain this collective, critical approach has been described as "something unusual".[12]

The following is a list of some of the criticisms commonly made of the Bourbaki approach:[13]

- algorithmic content is not considered on-topic and is almost completely omitted[14]
- problem solving, in the sense of heuristics, receives less emphasis than axiomatic theory-building[15]
- analysis is treated 'softly', without 'hard' estimates[16]
- Measure theory is developed from a functional analytic perspective. Taking the case of locally compact measure spaces as fundamental focuses the presentation on Radon measures and leads to an approach to measurable functions that is cumbersome, especially from the viewpoint of probability theory.[17] However, the last chapter of the book addresses limitations, especially for use in probability theory, of the restriction to locally compact spaces.
- combinatorics is not discussed
- logic is treated minimally[18]
- applications are not covered.

Furthermore, Bourbaki make only limited use of pictures in their presentation.[19] In general, Bourbaki has been criticized for reducing geometry as a whole to abstract algebra and soft analysis.[20]

Dieudonné as speaker for Bourbaki

Public discussion of, and justification for, Bourbaki's thoughts has in general been through Jean Dieudonné (who initially was the 'scribe' of the group) writing under his own name. In a survey of *le choix bourbachique* written in 1977, he did not shy away from a hierarchical development of the 'important' mathematics of the time.

He also wrote extensively under his own name: nine volumes on analysis, perhaps in belated fulfillment of the original project or pretext; and also on other topics mostly connected with algebraic geometry. While Dieudonné could reasonably speak on Bourbaki's encyclopedic tendency, and tradition (after innumerable frank *tais-toi, Dieudonné!* ("Hush, Dieudonné!") remarks at the meetings), it may be doubted whether all others agreed with him about mathematical writing and research. In particular Serre has often criticised the way the Bourbaki works were written, and has championed in France greater attention to problem-solving, within number theory especially, not an area treated in the main Bourbaki texts.

Dieudonné stated the view that most workers in mathematics were doing ground-clearing work, in order that a future Riemann could find the way ahead intuitively open. He pointed to the way the axiomatic method can be used as a tool for problem-solving, for example by → Alexander Grothendieck. Others found him too close to Grothendieck to be an unbiased observer. Comments in Pal Turán's 1970 speech on the award of a Fields Medal to Alan Baker about theory-building and problem-solving were a reply from the traditionalist camp at the next opportunity[21] , Grothendieck having received a Fields Medal *in absentia* in 1966 and the awards being every four years.

Bourbaki's influence on mathematics education

In the longer term, the manifesto of Bourbaki has had a definite and deep influence. In secondary education the new math movement corresponded to teachers influenced by Bourbaki. In France the change was secured by the Lichnerowicz Commission.[22]

The influence on graduate education in pure mathematics is perhaps most noticeable in the treatment now current of Lie groups and Lie algebras. Dieudonné at one point said 'one can do nothing serious without them', for which he was reproached; but the change in Lie theory to its everyday usage owes much to the type of exposition Bourbaki championed. Beforehand Jacques Hadamard despaired of ever getting a clear idea of it.

See also

- Bourbaki–Witt theorem
- Arthur Besse

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[1] *Confronted by the task of appraising a book by N. Bourbaki, this reviewer feels as if he were required to climb the Nordwand of the Eiger. The presentation is austere and monolithic. The route is beset by scores of definitions, many of them apparently unmotivated. Always there are hordes of exercises to be worked through painfully. One must be prepared to make constant cross-references to the author's many other works.* Hewitt, Edwin (1956). "Review: Espaces vectoriels topologiques". *Bulletin of the American Mathematical Society* **62**: 507–508. doi: 10.1090/S0002-9904-1956-10042-6 (http://dx.doi.org/10.1090/S0002-9904-1956-10042-6). (http://www.ams.org/bull/1956-62-05/S0002-9904-1956-10042-6/home.html)

[2] *...by 1958 when the original six books were completed, the first few of these books were already almost 20 years out of date.* (http://turnbull.mcs.st-and.ac.uk/~history/PrintHT/Bourbaki_2.html)

[3] (1) the symbols do not appear in Bourbaki publications (rather, ordinary bold is used) at or near the era when they began to be used elsewhere, for instance, in typewritten lecture notes from Princeton University (achieved in some cases by overstriking R or C with I), and (an apparent first) typeset in Gunning and Rossi's textbook on several complex variables; (2) Jean-Pierre Serre, a member of the Bourbaki group, has publicly inveighed against the use of "blackboard bold" anywhere other than on a blackboard.

[4] *Bourbaki came to terms with Poincaré only after a long struggle. When I joined the group in the fifties it was not the fashion to value Poincaré at all. He was old-fashioned.* Pierre Cartier interviewed by Marjorie Senechall. "The Continuing Silence of Bourbaki". *Mathematical Intelligencer* **19**: 22–28. 1998. (http://www.ega-math.narod.ru/Bbaki/Cartier.htm)

[5] Ian Stewart: *Mathematicians knew how to decode Bourbakist messages, but the rest of the world didn't. This led to unfortunate misunderstandings, and by the end of the sixties, mathematics and physics departments were no longer on speaking terms.* Ian Stewart (11 1995). "Bye-Bye Bourbaki: Paradigm Shifts in Mathematics". *The Mathematical Gazette* (The Mathematical Association) **79** (486): 496–498. doi: 10.2307/3618076 (http://dx.doi.org/10.2307/3618076).

[6] Borel (1998)

[7] Chevalley in Guedj (1985)

[8] The minutes are in the Bourbaki archives — for a full description of the initial meeting consult Liliane Beaulieu in the *Mathematical Intelligencer*.

[9] This resulted in a complete change of personnel by 1958; see Robert Mainard paper cited below. However, the Aubin paper cited below quotes the historian Liliane Beaulieu as never having found written affirmation of this rule.

[10] Charles Denis Bourbaki fought in the Crimean War and Franco-Prussian War, refer to A. Weil: The Apprenticeship of a Mathematician, Birkhäuser Verlag 1992, pp 93-122.

[11] It is said that Weil's wife Evelyne supplied *Nicolas*. (Mentioned by McCleary (PDF) (http://www.math.vassar.edu/faculty/mccleary/Bourbaki.pdf). This is more or less confirmed by Robert Mainard (http://www.academie-stanislas.org/TomeXIII/Mainard98.pdf)((PDF), a long article in French, which gives numerous further details: why N?, and the prank lecture of Raoul Husson in a false beard that gave rise to *Bourbaki's theorem*). They married in 1937, she having previously been with de Possel; who then unsurprisingly left the group.

[12] Hector C. Sabelli, Louis H. Kauffman, BIOS (2005), p. 423.

[13] Pierre Cartier, a Bourbaki member 1955–1983, comments explicitly on several of these points (*The Continuing Silence of Bourbaki*, article from the *Mathematical Intelligencer* (http://ega-math.narod.ru/Bbaki/Cartier.htm)): *...essentially no analysis beyond the foundations: nothing about partial differential equations, nothing about probability. There is also nothing about combinatorics, nothing about algebraic topology, nothing about concrete geometry. And Bourbaki never seriously considered logic. Dieudonné himself was very vocal against logic. Anything connected with mathematical physics is totally absent from Bourbaki's text.*

[14] This is one of the reasons for diminishing influence: *Le développement des mathématiques dites appliquées, de la statistique et des probabilités, des théories liées à l'informatique a diminué l'influence de Bourbaki* (http://publimath.irem.univ-mrs.fr/glossaire/BO025.htm)

[15] Tim Gowers discusses at length the *distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories* in his *The Two Cultures of Mathematics* (PDF) (http://www.dpmms.cam.ac.uk/~wtg10/2cultures.pdf).

[16] Lennart Carleson spoke of this in an interview (*Infomat* August 2006 (PDF) (http://www.matematikkforeningen.no/INFOMAT/06/0608.pdf)): *...that book [from 1968] was written mostly as a way to encourage the teachers to stay with established values. That was during the Bourbaki and New Math period and mathematics was really going to pieces, I think. The teachers were very worried and they had very little backing.*

[17] Heinz König: *The traditional abstract measure theory which emerged from the achievements of Borel and Lebesgue in the first two decades of the 20th century is burdened with its total limitation to sequential procedures and its neglect of regularity. The alternative theory due to Bourbaki which arose in the middle of the century was able to relieve these burdens, but produced new ones. In particular its fundamental turn to inner regularity, based on the profound role of compactness, was done with the inappropriate weapons from the outer arsenal, which subsequently enforced that unfortunate construction named the essential one. All this produced serious obstacles against a unified theory of measure and integration, for example for the notion of signed measures, the formation of products and for the representation theorems of Daniell-Stone and Riesz types.* (http://www.math.tu-dresden.de/~pos_iv/Abstracts/koenig_abstract/index.html)

[18] Discussed by the set theorist Adrian Mathias (*The Ignorance of Bourbaki* (PDF) (http://www.dpmms.cam.ac.uk/~ardm/bourbaki.pdf)). See also Mashaal (2006), p.120, "Lack of interest in foundations".

[19] Pierre Cartier, in the article cited above, is quoted as later saying *The Bourbaki were Puritans, and Puritans are strongly opposed to pictorial representations of truths of their faith.*

[20] In the French context it has been said that geometry was in effect exiled from secondary teaching: *Pour ce qui est des années 1960, l'effet de la réforme dite des mathématiques modernes sur l'enseignement de la géométrie est bien connu : si Dieudonné, comme Bourlet finalement, lance "A bas Euclide", le résultat n'est pas l'élaboration d'une géométrie plus expérimentale, plus intuitive. C'est l'effacement de la géométrie derrière l'algèbre linéaire et la quasi-disparition de l'enseignement de la géométrie élémentaire au collège et au lycée pour une dizaine d'années.*—"As for the 1960s, the effect of this reform of modern mathematics on the teaching of geometry is well-known: if Dieudonné, like Bourlet finally, says "push Euclid back," the result is not the development of a geometry that is more experimental, more intuitive. It's the erasure of geometry behind linear algebra, and the quasi-disappearance of the teaching of elementary geometry in high school, for ten years." (http://www.apmep.asso.fr/spip.php?article210)

[21] On the Work of Alan Baker (http://links.jstor.org/sici?sici=0022-4812(197209)37:3<606:OTWOAB>2.0.CO;2-9)

[22] Mashaal (2006) Ch.10: New Math in the Classroom

[23] http://www.ams.org/notices/199803/borel.pdf

[24] http://dx.doi.org/10.1007%2FBF03024169

[25] http://www.institut.math.jussieu.fr/~daubin/publis/1997.pdf

[26] http://www.bourbaki.ens.fr/

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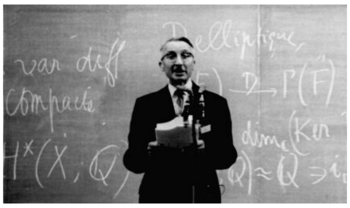
[28] http://www.ega-math.narod.ru/Bbaki/Bourb3.htm

[29] http://www-history.mcs.st-andrews.ac.uk/Biographies/Bourbaki.html

Henri Cartan

Henri Cartan	
	
Born	July 8, 1904 Nancy, France
Died	August 13, 2008 (aged 104) Paris, France
Nationality	French
Occupation	Mathematician
Spouse(s)	Nicole Antoinette Weiss
Parents	Élie Cartan Marie-Louise Bianconi

Henri Cartan	
Fields	→ Algebraic topology Bourbaki
Doctoral advisor	Paul Montel
Doctoral students	Jean-Paul Benzécri Jean-Paul Brasselet Pierre Cartier Jean Cerf Jacques Deny Adrien Douady Roger Godement Max Karoubi Jean-Louis Koszul Joshua Leslie Jean-Pierre Ramis → Jean-Pierre Serre Banwari Sharma → René Thom

Henri Paul Cartan (July 8, 1904 – August 13, 2008) was a son of Élie Cartan, and was, as his father was, a distinguished and influential French mathematician.[1]

Life

Cartan studied at the Lycée Hoche in Versailles, then at the ENS, receiving his doctorate in mathematics. He taught at the University of Strasbourg from November 1931 until the outbreak of the Second World War, after which he held academic positions at a number of other French universities, spending the bulk of his working life in Paris.

Cartan was known for work in → algebraic topology, in particular on cohomology operations, Killing homotopy groups and group cohomology. His seminar in Paris in the years after 1945 covered ground on several complex variables, sheaf theory, spectral sequences and → homological algebra, in a way that deeply influenced → Jean-Pierre Serre, Armand Borel, → Alexander Grothendieck and → Frank Adams, amongst others of the leading lights of the younger generation. The number of his official students was small, but includes Adrien Douady, Roger Godement, Max Karoubi, → Jean-Pierre Serre and → René Thom.

Cartan also was a founding member of the → Bourbaki group and one of its most active participants. His book with → Samuel Eilenberg *Homological Algebra* (1956)[2] was an important text, treating the subject with a moderate level of abstraction and category theory.

Cartan used his influence to help obtain the release of some dissident mathematicians, including Leonid Plyushch and Jose Luis Massera. For his humanitarian efforts he received the Pagels Award from the New York Academy of Sciences.[3]

Cartan died on 13 August, 2008 at the age of 104. His funeral took place the following Wednesday on 20 August in Die, Drome.[1]

Honours and awards

Cartan received numerous honours and awards including the Wolf Prize in 1980. From 1974 until his death he had been a member of the French Academy of Sciences. He was a foreign member of the Finnish Academy of Science and Letters, Royal Danish Academy of Sciences and Letters, Royal Society of London, Russian Academy of Sciences, Royal Swedish Academy of Sciences, United States National Academy of Sciences, Polish Academy of Sciences and other academies and societies.

See also

- Cartan's theorems A and B

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[2] Henri Cartan and Samuel Eilenberg, *Homological Algebra* ISBN 978-0-691-04991-5
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Samuel Eilenberg

Samuel Eilenberg	
 Samuel Eilenberg (1970)	
Born	September 30, 1913 Warsaw, Russian Empire
Died	January 30, 1998 (aged 84)
Institutions	Columbia University
Alma mater	Warsaw University
Doctoral advisor	→ Karol Borsuk
Known for	Eilenberg-Steenrod axioms Eilenberg swindle

Samuel Eilenberg (September 30, 1913—January 30, 1998) was a Polish and American mathematician of Jewish descent. He was born in Warsaw, Russian Empire (now in Poland) and died in New York City, USA, where he had spent much of his career as a professor at Columbia University.

He earned his Ph.D. from Warsaw University in 1936. His thesis advisor was → Karol Borsuk. His main interest was → algebraic topology. He worked on the axiomatic treatment of → homology theory with Norman Steenrod (whose names the Eilenberg-Steenrod axioms bear), and on → homological algebra with → Saunders Mac Lane. In the process, Eilenberg and Mac Lane created category theory.

Eilenberg took part in the → Bourbaki group meetings, and, with → Henri Cartan, wrote the 1956 book *Homological Algebra*, which became a classic.

Later in life he worked mainly in pure category theory, being one of the founders of the field. The Eilenberg swindle (or *telescope*) is a construction applying the telescoping cancellation idea to projective modules.

Eilenberg also wrote an important book on automata theory. The X-machine, a form of automaton, was introduced by Eilenberg in 1974.

Eilenberg was also a prominent collector of Asian art. His collection mainly consisted of small sculptures and other artifacts from India, Indonesia, Pakistan, Nepal, Thailand, Cambodia, Sri Lanka and Central Asia. In 1991-1992, the Metropolitan Museum of Art in New York staged an exhibition from more than 400 items that Eilenberg had donated to the museum, entitled *The Lotus Transcendent: Indian and Southeast Asian Art From the Samuel Eilenberg Collection"*.[1]

Selected publications

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See also

- Stefan Banach
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- Eilenberg–MacLane space
- Eilenberg–Moore spectral sequence

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Peter Freyd

Peter J. Freyd is an American mathematician, a professor at the University of Pennsylvania, known for work in category theory and for founding the False Memory Syndrome Foundation.

Mathematical work

Freyd is perhaps best known as the author of the foundational book *Abelian Categories: An Introduction to the Theory of Functors*. This work culminates in a proof of Mitchell's embedding theorem.

False Memory Syndrome Foundation

Freyd and his wife Pamela founded the False Memory Syndrome Foundation in 1992,[1] after Freyd was accused of sexual abuse by his daughter Jennifer.[1] [2] Freyd denies the accusations.[3]

CV

- Ph.D. Princeton University 1960.

Publications

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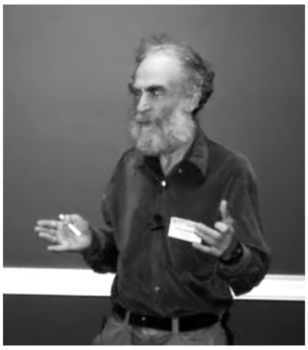
[5] http://www.emis.de/journals/TAC/reprints/articles/3/tr3abs.html

Mikhail Gromov

Mikhail Leonidovich Gromov	
 Mikhail Gromov	
Born	23 December 1943 Boksitogorsk, Russian SFSR
Residence	France
Nationality	Franco-Russian
Fields	Mathematician
Institutions	IHÉS New York University
Alma mater	Leningrad (PhD)
Doctoral advisor	V. A. Rokhlin
Known for	geometry
Notable awards	see text

Mikhail Leonidovich Gromov (Russian: Михаил Леонидович Громов; born 23 December 1943) also romanized as Mikhael Gromov or Michael Gromov) is a Franco-Russian mathematician known for important contributions in many different areas of mathematics. He is considered a geometer in a very broad sense of the word.

Work

His style of geometry features a "coarse" or "soft" viewpoint, often analyzing asymptotic or large-scale properties.

Gromov's impact has been felt most heavily in geometric group theory, where he characterized groups of polynomial growth and created the notion of hyperbolic group; symplectic topology, where he introduced pseudoholomorphic curves, and in Riemannian geometry. His work, however, has delved deeply into analysis and algebra, where he will often formulate a problem in "geometric" terms. For example, his h-principle on differential relations is the basis for a geometric theory of partial differential equations.

Gromov studied for a doctorate (1973) in Leningrad, where he was a student of V. A. Rokhlin. He is now a permanent member of IHÉS, and a Professor of Mathematics at New York University.

Prizes and honors

Prizes

- Prize of the Mathematical Society of Moscow (1971)
- Oswald Veblen Prize in Geometry (AMS) (1981)
- Prix Elie Cartan de l'Academie des Sciences de Paris (1984)
- Prix de l'Union des Assurances de Paris (1989)
- Wolf Prize in Mathematics (1993)
- Leroy P. Steele Prize for Seminal Contribution to Research (AMS) (1997)
- Lobachevsky Medal (1997)
- Balzan Prize for Mathematics (1999)
- Kyoto Prize in Mathematical Sciences (2002)
- Nemmers Prize in Mathematics (2004)
- Bolyai prize in 2005
- Abel prize in 2009 “for his revolutionary contributions to geometry”

Honors

- Invited speaker to International Congress of Mathematicians: 1970 (Nice), 1978 (Helsinki), 1982 (Warsaw), 1986 (Berkeley)
- Foreign member of the National Academy of Sciences and American Academy of Arts and Sciences
- Membre de l'Institut de France - Académie des Sciences

See also

- Gromov's theorem on groups of polynomial growth
- Gromov's theorem on almost flat manifolds
- Gromov's compactness theorem
- Gromov's inequality for complex projective space
- Gromov's systolic inequality for essential manifolds
- Gromov–Hausdorff convergence
- Bishop–Gromov inequality
- Lévy-Gromov inequality
- Gromov-Witten invariants
- Taubes's Gromov invariant
- Minimal volume
- localisation on the sphere
- Gromov norm
- Hyperbolic group
- Random group
- Ramsey-Dvoretzky-Milman phenomenon
- Systolic geometry
- Filling radius
- Gromov product
- Gromov δ-hyperbolic space
- Filling area conjecture

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External links

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[6] http://genealogy.math.ndsu.nodak.edu/id.php?id=14999

Alexander Grothendieck

Alexander Grothendieck in Montreal, 1970

Born	March 28, 1928 Berlin, Germany
Residence	France
Nationality	Stateless
Field	Mathematician
Academic advisor	Laurent Schwartz
Notable students	Pierre Deligne, Jean-Louis Verdier, Michel Raynaud
Known for	algebraic geometry, → homological algebra, and functional analysis
Notable prizes	Fields Medal (1966), Crafoord Prize(1988, declined)
edit data [1]	

Alexander Grothendieck (born March 28, 1928 in Berlin, Germany) is considered one of the greatest mathematicians of the 20th century.[2] [3] He is most famous for his revolutionary advances in algebraic geometry, but he has also made major contributions to → algebraic topology, number theory, category theory, Galois theory, descent theory, commutative → homological algebra and functional analysis. He was awarded the Fields Medal in 1966, and was co-awarded the Crafoord Prize with Pierre Deligne in 1988. He declined the latter prize on ethical grounds in an open letter to the media.

He is noted for his mastery of abstract approaches to mathematics, and his perfectionism in matters of formulation and presentation. In particular, he demonstrated the ability to derive concrete results using only very general methods.[4] [5] [6] Relatively little of his work after 1960 was published by the conventional route of the learned journal, circulating initially in duplicated volumes of seminar notes; his influence was to a considerable extent personal, on French mathematics and the Zariski school at Harvard University. He is the subject of many stories and some misleading rumors concerning his work habits and politics, his confrontations with other mathematicians and the French authorities, his withdrawal from mathematics at age 42, his retirement, and his subsequent lengthy writings.

Mathematical achievements

Grothendieck's early mathematical work was done in functional analysis between 1949 and 1953 working on (what became) his doctoral thesis in Nancy, supervised by Jean Dieudonné and Laurent Schwartz. His key contributions include topological tensor products of vector spaces, the theory of nuclear spaces and the application of Lp spaces in studying linear maps between topological vector spaces. In the space of a few years, he had turned himself into a leading authority on the theory of topological vector spaces — to the extent that Dieudonné compares his impact in this field to that of Banach [7] .

However, it is algebraic geometry and related fields where Grothendieck did his most important and influential work. From about 1955 he started to work on sheaf theory and → homological algebra, rapidly producing the very influential "Tôhoku paper" (*Sur quelques points d'algèbre homologique*, published in 1957) where he introduced Abelian categories and applied it to show that sheaf cohomology can be defined as certain derived functors in this context.

Homological methods and sheaf theory had already been introduced in algebraic geometry by → Jean-Pierre Serre and others, after sheaves had been defined by Jean Leray. Grothendieck took them to a higher level of abstraction and turned them into a key organising principle of his theory. He thereby changed the tools and the level of abstraction in algebraic geometry. He shifted attention from the study of individual varieties to the *relative point of view* (pairs of varieties related by a morphism), allowing a broad generalization of many classical theorems. The first major application was the relative version of → Serre's theorem showing that the cohomology of a coherent sheaf on a complete variety is finite dimensional; Grothendieck's theorem shows that the higher direct images of coherent sheaves under a proper map are coherent; this reduces to Serre's theorem over a one-point space.

Next, in 1956, he applied the same thinking to the Riemann–Roch theorem, which had already recently been generalized to any dimension by Hirzebruch. The Grothendieck–Riemann–Roch theorem was announced by Grothendieck at the initial Mathematische Arbeitstagung in Bonn, in 1957. It appeared in print in a paper written by Armand Borel with Serre. This result was his first major achievement in algebraic geometry. He went on to plan and execute a major foundational programme for rebuilding the foundations of algebraic geometry; he exposed the main outlines of this programme in his talk at the 1958 International Congress of Mathematicians.

His foundational work on algebraic geometry is at a higher level of abstraction than all prior versions. He adapted the use of non-closed generic points, which led to the theory of schemes. He also pioneered the systematic use of nilpotents. As 'functions' these can take only the value 0, but they carry infinitesimal information, in purely algebraic settings. His *theory of schemes* has become established as the best universal foundation for this major field, because of its great expressive power as well as technical depth. In that setting one can use birational geometry, techniques from number theory, Galois theory and commutative algebra, and close analogues of the methods of → algebraic topology, all in an integrated way.

His influence spilled over into many other branches of mathematics, for example the contemporary theory of D-modules. (It also provoked adverse reactions, with many mathematicians seeking out more concrete areas and problems.)

EGA and SGA

The bulk of Grothendieck's published work is collected in the monumental, and yet incomplete, *Éléments de géométrie algébrique* (EGA) and *Séminaire de géométrie algébrique* (SGA). The collection *Fondements de la Géometrie Algébrique* (FGA), which gathers together talks given in the Séminaire Bourbaki, also contains important material.

Perhaps Grothendieck's deepest single accomplishment is the invention of the étale and l-adic cohomology theories, which explain an observation of André Weil's that there is a deep connection between the topological characteristics of a variety and its diophantine (number theoretic) properties. For example, the number of solutions of an equation over a finite field reflects the topological nature of its solutions over the complex numbers. Weil realized that to prove such a connection one needed a new cohomology theory, but neither he nor any other expert saw how to do this until such a theory was found by Grothendieck.

This program culminated in the proofs of the Weil conjectures, the last of which was settled by Grothendieck's student Pierre Deligne in the early 1970s after Grothendieck had largely withdrawn from mathematics.

Major mathematical topics (from *Récoltes et Semailles*)

He wrote a retrospective assessment of his mathematical work (see the external link *La Vision* below). As his main mathematical achievements ("maître-thèmes"), he chose this collection of 12 topics (his chronological order):

1. Topological tensor products and nuclear spaces
2. "Continuous" and "discrete" duality (derived categories and "six operations").
3. *Yoga* of the Grothendieck–Riemann–Roch theorem (→ K-theory, relation with intersection theory).
4. Schemes.
5. Topoi.
6. Étale cohomology including l-adic cohomology.
7. Motives and the motivic Galois group (and Grothendieck categories)
8. Crystals and crystalline cohomology, *yoga* of De Rham and Hodge coefficients.
9. Topological algebra, infinity-stacks, 'dérivateurs', cohomological formalism of toposes as an inspiration for a new homotopic algebra
10. Tame topology.
11. *Yoga* of anabelian geometry and Galois–Teichmüller theory.
12. Schematic point of view, or "arithmetics" for regular polyhedra and regular configurations of all sorts.

He wrote that the central theme of the topics above is that of topos theory, while the first and last were of the least importance to him.

Here the term *yoga* denotes a kind of "meta-theory" that can be used heuristically. The word *yoke*, meaning "linkage", is derived from the same Indo-European root.

Life

Family and early life

Alexander Grothendieck was born in Berlin to anarchist parents: a Russian father from an ultimately Hassidic family, Alexander Shapiro aka Tanaroff, and a mother from a German Protestant family, Johanna "Hanka" Grothendieck; both of his parents had broken away from their early backgrounds in their teens[8] . At the time of his birth Grothendieck's mother was married to Johannes Raddatz, a German journalist, and his birthname was initially recorded as *Alexander Raddatz*. The marriage was dissolved in 1929 and Shapiro/Tanaroff acknowledged his paternity, but never married Hanka Grothendieck[9] . Grothendieck lived with his parents until 1933 in Berlin. At the end of that year, Shapiro moved to Paris, and Hanka followed him the next year. They left Grothendieck in the care of Wilhelm Heydorn, a Lutheran Pastor and teacher [10] in Hamburg where he went to school. During this time, his parents fought in the Spanish Civil War.

During WWII

In 1939 Grothendieck came to France and lived in various camps for displaced persons with his mother, first at the Camp de Rieucros, spending 1942–44 at Le Chambon-sur-Lignon. His father was sent via Drancy to Auschwitz where he died in 1942.

Studies and contact with research mathematics

After the war, the young Grothendieck studied mathematics in France, initially at the University of Montpellier. He had decided to become a math teacher because he had been told that mathematical research had been completed early in the 20th century and there were no more open problems.[11] However, his talent was noticed, and he was encouraged to go to Paris in 1948.

Initially, Grothendieck attended → Henri Cartan's Seminar at École Normale Supérieure, but lacking the necessary background to follow the high-powered seminar, he moved to the University of Nancy where he wrote his dissertation under Laurent Schwartz in functional analysis, from 1950 to 1953. At this time he was a leading expert in the theory of topological vector spaces. By 1957, he set this subject aside in order to work in algebraic geometry and → homological algebra.

The IHÉS years

Installed at the Institut des Hautes Études Scientifiques (IHÉS), Grothendieck attracted attention by an intense and highly productive activity of seminars (*de facto* working groups drafting into foundational work some of the ablest French and other mathematicians of the younger generation). Grothendieck himself practically ceased publication of papers through the conventional, learned journal route. He was, however, able to play a dominant role in mathematics for around a decade, gathering a strong school.

During this time he had officially as students Michel Demazure (who worked on SGA3, on group schemes), Luc Illusie (cotangent complex), Michel Raynaud, Jean-Louis Verdier (cofounder of the derived category theory) and Pierre Deligne. Collaborators on the SGA projects also included Mike Artin (étale cohomology) and Nick Katz (monodromy theory and Lefschetz pencils). Jean Giraud worked out torsor theory extensions of non-abelian cohomology. Many others were involved.

The 'Golden Age'

Alexander Grothendieck's work during the `Golden Age' period at IHÉS established several unifying themes in algebraic geometry, number theory, → topology, category theory and complex analysis. His first (pre-IHÉS) breakthrough in algebraic geometry was the Grothendieck–Hirzebruch–Riemann–Roch theorem, a far-reaching generalisation of the Hirzebruch–Riemann–Roch theorem proved algebraically; in this context he also introduced → K-theory. Then, following the programme he outlined in his talk at the 1958 International Congress of Mathematicians, he introduced the theory of schemes, developing it in detail in his Éléments de géométrie algébrique (EGA) and providing the new more flexible and general foundations for algebraic geometry that has been adopted in the field since that time. He went on to introduce the étale cohomology theory of schemes, providing the key tools for proving the Weil conjectures, as well as crystalline cohomology and algebraic de Rham cohomology to complement it. Closely linked to these cohomology theories, he originated topos theory as a generalisation of topology (relevant also in mathematical logic, category theory, and also to computer software/programming and institutional ontology classification and bioinformatics). He also provided an algebraic definition of → fundamental groups of schemes and more generally the main structures of a categorical Galois theory. As a framework for his coherent duality theory he also introduced derived categories, which were further developed by Verdier. The results of work on these and other topics were published in the EGA and in less polished form in the notes of the Séminaire de géométrie algébrique (SGA) that he directed at IHES.

Politics and retreat from scientific community

Grothendieck's political views were radical and pacifist, but not communist (thus he strongly disapproved of the Soviet military expansionism as well). He gave lectures on category theory in the forests surrounding Hanoi while the city was being bombed, to protest against the Vietnam War (*The Life and Work of Alexander Grothendieck*, American Math. Monthly, vol. 113, no. 9, footnote 6). He retired from scientific life around 1970, after having discovered the partly military funding of IHÉS (see pp. xii and xiii of SGA1, Springer Lecture Notes 224). He returned to academia a few years later as a professor at the University of Montpellier, where he stayed until his retirement in 1988. His criticisms of the scientific community, and especially of several mathematics circles, are also contained in a letter [12], written in 1988, in which he states the reasons for his refusal of the Crafoord Prize.

While the issue of military funding was perhaps the most obvious explanation for Grothendieck's departure from IHÉS, those who knew him say that the causes of the rupture ran deeper. Pierre Cartier, a visiteur de longue durée at the IHÉS, wrote a piece about Grothendieck for a special volume published on the occasion of the IHÉS's fortieth anniversary. The *Grothendieck Festschrift* was a three-volume collection of research papers to mark his sixtieth birthday (falling in 1988), and published in 1990.[13]

In it Cartier notes that, as the son of an antimilitary anarchist and one who grew up among the disenfranchised, Grothendieck always had a deep compassion for the poor and the downtrodden. As Cartier puts it, Grothendieck came to find Bures-sur-Yvette "une cage dorée" [a golden cage]. While Grothendieck was at the IHÉS, opposition to the Vietnam War was heating up, and Cartier suggests that this also reinforced Grothendieck's distaste at having become a mandarin of the scientific world. In addition, after several years at the IHÉS Grothendieck seemed to cast about for new intellectual interests. By the late 1960s he had started to become interested in scientific areas outside of mathematics. David Ruelle, a physicist who joined the IHÉS faculty in 1964, said that Grothendieck came to talk to him a few times about physics. (In the 1970s Ruelle and the Dutch mathematician Floris Takens produced a new model for turbulence, and it was Ruelle who invented the concept of a strange attractor in a dynamical system.) Biology interested Grothendieck much more than physics, and he organized some seminars on biological topics.[14] After leaving the IHÉS, Grothendieck tried but failed to get a position at the Collège de France. He then went to Université de Montpellier, where he became increasingly estranged from the mathematical community. Around this time, he founded a group called Survivre, which was dedicated to antimilitary and ecological issues. His mathematical career, for the most part, ended when he left the IHÉS. In 1984 he wrote a proposal to get a position through the Centre National de la Recherche Scientifique. The proposal, entitled Esquisse d'un Programme [Sketch of a Program] describes new ideas for studying the moduli space of complex curves. Although Grothendieck himself never published his work in this area, the proposal became the inspiration for work by other mathematicians and the source of the theory of dessins d’enfants. Esquisse d’un Programme was published in the two-volume proceedings Geometric Galois Actions (Cambridge University Press, 1997)."[15]

Manuscripts written in the 1980s

While not publishing mathematical research in conventional ways during the 1980s, he produced several influential manuscripts with limited distribution, with both mathematical and biographical content. During that period he also released his work on Bertini type theorems contained in EGA 5, published by the Grothendieck Circle [16] in 2004.

La Longue Marche à travers la théorie de Galois [*The Long March Through Galois Theory*] is an approximately 1600-page handwritten manuscript produced by Grothendieck during the years 1980–1981, containing many of the ideas leading to the *Esquisse d'un programme*[17] (see below, and also a more detailed entry[18]), and in particular studying the Teichmüller theory. (For an English translation of the tables of contents of these manuscripts see the Wikipedia separate entry on the Esquisse d'un programme.)

In 1983 he wrote a huge extended manuscript (about 600 pages) entitled *Pursuing Stacks*, stimulated by correspondence with → Ronald Brown, (see also R.Brown [19] and Tim Porter [20] at University of Bangor in Wales), and starting with a letter addressed to → Daniel Quillen. This letter and successive parts were distributed from Bangor (see External Links below): in an informal manner, as a kind of diary, Grothendieck explained and developed his ideas on the relationship between algebraic homotopy theory and algebraic geometry and prospects for a noncommutative theory of stacks. The manuscript, which is being edited for publication by G. Maltsiniotis, later led to another of his monumental works, *Les Dérivateurs*. Written in 1991, this latter opus of about 2000 pages further developed the homotopical ideas begun in *Pursuing Stacks*. Much of this work anticipated the subsequent development of the motivic homotopy theory of F. Morel and V. Voevodsky in the mid 1990s.

His *Esquisse d'un programme*[17] (1984) is a proposal for a position at the Centre National de la Recherche Scientifique, which he held from 1984 to his retirement in 1988. Ideas from it have proved influential, and have been developed by others, in particular dessins d'enfants and a new field emerging as anabelian geometry. In *La Clef des*

Songes he explains how the reality of dreams convinced him of God's existence.

The 1000-page autobiographical manuscript *Récoltes et semailles* (1986) is now available on the internet in the French original, and an English translation is underway (these parts of Récoltes et semailles have already been translated into Russian [21] and published in Moscow). Some parts of *Récoltes et semailles*[22] [23] and the whole *La Clef des Songes* [24] have been translated into Spanish.

Disappearance

In 1991, Grothendieck left his home and disappeared. He is now said to live in southern France or Andorra and to entertain no visitors. Though he has been inactive in mathematics for many years, he remains one of the greatest and most influential mathematicians of modern times.

See also

- Ax-Grothendieck theorem
- Birkhoff–Grothendieck theorem
- Esquisse d'un Programme
- Grothendieck category [25]
- Grothendieck's connectedness theorem
- Grothendieck connection
- Grothendieck construction
- Grothendieck's Galois theory
- Grothendieck group
- Grothendieck inequality or Grothendieck constant
- Grothendieck–Katz p-curvature conjecture
- Grothendieck's relative point of view
- Grothendieck–Riemann–Roch theorem
- Grothendieck's Séminaire de géométrie algébrique
- Grothendieck space
- Grothendieck spectral sequence
- → Grothendieck topology
- Grothendieck universe
- Tarski–Grothendieck set theory
- IHES
- IHES at Forty by Allyn Jackson [26]

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External links

- O'Connor, John J.; Robertson, Edmund F., "Alexander Grothendieck [37]", *MacTutor History of Mathematics archive*.
- Alexander Grothendieck [38] at the Mathematics Genealogy Project
- Grothendieck Circle [16], collection of mathematical and biographical information, photos, links to his writings
  - Grothendieck Circle discussion Forum [39]
- Institut des Hautes Études Scientifiques [40]
- The origins of `Pursuing Stacks' [41] This is an account of how `Pursuing Stacks' was written in response to a correspondence in English with Ronnie Brown and Tim Porter at Bangor, which continued until 1991.
- Récoltes et Semailles [41] in French.
- Spanish translation [42] of "Récoltes et Semailles" et "Le Clef des Songes" and other Grothendieck's texts
- short bio [34] from Notices of the American Mathematical Society

Persondata

NAME	Grothendieck, Alexander
ALTERNATIVE NAMES	
SHORT DESCRIPTION	Mathematician
DATE OF BIRTH	March 28, 1928
PLACE OF BIRTH	Berlin, Germany
DATE OF DEATH	,
PLACE OF DEATH	

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[8] Society for Industrial and Applied Mathematics (http://www.siam.org/news/news.php?id=1405)

[9] Society for Industrial and Applied Mathematics (http://www.siam.org/news/news.php?id=1405)

[10] Allyn Jackson, The Life of Alexander Grothendieck, p. 1040 (http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf)

[11] See Jackson (2004:1). The remark is from the beginning of Récoltes et Semailles (page P4, in the introductory section Prélude en quatre Mouvements)

[12] http://web.archive.org/web/20060106062005/http://www.math.columbia.edu/~lipyan/CrafoordPrize.pdf

[13] The editors were Pierre Cartier, Luc Illusie, Nick Katz, Gérard Laumon, Yuri Manin, and Ken Ribet. A second edition has been printed (2007) by Birkhauser.

[14] http://www.ams.org/notices/199903/ihes-changes.pdf

[15] http://books.google.co.uk/books?id=Hlf1j2XmXkcC&dq=Geometric+Galois+Actions&printsec=frontcover&source=bl&ots=q9iN4QMkYj&sig=fAsvVlPOhN9K9LfjGFJC5zm_IA4&hl=en&ei=KW-3ScSkHYHIMuDukOEK&sa=X&oi=book_result&resnum=1&ct=result

[16] http://www.math.jussieu.fr/~leila/grothendieckcircle/index.php

[17] http://matematicas.unex.es/~navarro/res/esquissefr.pdf

[18] http://en.wikipedia.org/wiki/Esquisse_d%27un_Programme

[19] http://www.bangor.ac.uk/r.brown

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Heinz Hopf

Heinz Hopf (on the right) in Oberwolfach, together with Hellmuth Kneser

Heinz Hopf (November 19, 1894 – June 3, 1971) was a German mathematician born in Gräbschen, Germany (now Grabiszyn, part of Wrocław, Poland). He attended Dr. Karl Mittelhaus' higher boys' school from 1901 to 1904, and then entered the König-Wilhelm-Gymnasium in Breslau. He showed mathematical talent from an early age. In 1913 he entered the Silesian Friedrich Wilhelm University where he attended lectures by Ernst Steinitz, Kneser, Max Dehn, Erhard Schmidt, and Rudolf Sturm. When World War I broke out in 1914, Hopf eagerly enlisted. He was wounded twice and received the iron cross (first class) in 1918.

In 1920, Hopf moved to Berlin to continue his mathematical education. He studied under Ludwig Bieberbach, receiving his doctorate in 1925. In his dissertation, *Connections between topology and metric of manifolds* (German *Über Zusammenhänge zwischen Topologie und Metrik von Mannigfaltigkeiten*), he proved that any simply connected complete Riemannian 3-manifold of constant curvature is globally isometric to Euclidean, spherical, or hyperbolic space. He also studied the indices of zeros of vector fields on hypersurfaces, and connected their sum to curvature. Some six months later he gave a new proof that the sum of the indices of the zeros of a vector field on a manifold is independent of the choice of vector field and equal to the Euler characteristic of the manifold. This theorem is now called the Poincaré-Hopf theorem.

Hopf spent the year after his doctorate at Göttingen, where David Hilbert, Richard Courant, Carl Runge, and Emmy Noether were working. While there he met Paul Alexandrov and began a lifelong friendship.

In 1926 Hopf moved back to Berlin, where he gave a course in combinatorial topology. He spent the academic year 1927/28 at Princeton University on a Rockefeller fellowship with Alexandrov. Solomon Lefschetz, Oswald Veblen and J.W. Alexander were all at Princeton at the time. At this time Hopf discovered the Hopf invariant of maps $S^3 \to S^2$. and proved that the Hopf fibration has invariant 1. In the summer of 1928 Hopf returned to Berlin and began working with Alexandrov, at the suggestion of Courant, on a book on → topology. Three volumes were planned, but only one was finished. It was published in 1935.

In October 1928 Hopf married Anja von Mickwitz (1891 - 1967). The next year he declined a job offer from Princeton. In 1931 Hopf took Hermann Weyl's position at ETH, in Zürich.

Hopf received another invitation to Princeton in 1940, but he declined it. Two years later, however, he was forced to file for Swiss citizenship after his property was confiscated by Nazi authorities.

In 1946/47 and 1955/56 Hopf visited the United States, staying at Princeton and giving lectures at New York University and Stanford University. He served as president of the International Mathematical Union from 1955 to 1958. He received honorary doctorates from Princeton, Freiburg i. Br., Manchester, Sorbonne at Paris, Brussels, and Lausanne.

In memory of Hopf, ETH Zürich awards the Heinz Hopf Prize for *outstanding scientific work in the field of pure mathematics*.

See also

- Hopf algebra
- Hopf bifurcation (actually was done by Eberhard Hopf)
- Hopf bundle
- Hopf conjecture
- Hopf link
- H-space
- Hopf–Rinow theorem

External links

- O'Connor, John J.; Robertson, Edmund F., "Heinz Hopf [1]", *MacTutor History of Mathematics archive*.
- Heinz Hopf [2] at the Mathematics Genealogy Project

Academic Genealogy	
Notable teachers	**Notable students**
Ludwig Bieberbach Erhard Schmidt	Beno Eckmann Hans Freudenthal Werner Gysin Friedrich Hirzebruch Heinz Huber Michel Kervaire Willi Rinow Hans Samelson Eduard Stiefel

References

[1] http://www-history.mcs.st-andrews.ac.uk/Biographies/Hopf.html
[2] http://genealogy.math.ndsu.nodak.edu/id.php?id=17409

Michael J.Hopkins

Michael Jerome Hopkins (born April 18, 1958) is an American mathematician. He received his Ph.D. from Northwestern University in 1984 under the direction of Mark Mahowald. In 1984 he also received his D.Phil. from the University of Oxford under the supervision of Ioan James. He has been professor of mathematics at Harvard University since 2005, after fifteen years at MIT, a few years of teaching at Princeton University, a one-year position with the University of Chicago, and a visiting lecturer position at Lehigh University. He gave invited addresses at the 1990 Winter Meeting of the American Mathematical Society in Louisville, Kentucky, and at the 1994 International Congress of Mathematicians in Zurich. He presented the 1994 Everett Pitcher Lectures at Lehigh University, the 2000 Namboodiri Lectures at the University of Chicago, and the 2000 Marston Morse Memorial Lectures at the Institute for Advanced Study, Princeton. In 2001 he was awarded the Oswald Veblen Prize in Geometry from the AMS for his work in → homotopy theory. On 21 April 2009 Hopkins announced the solution of the Kervaire invariant problem, in joint work with Mike Hill and Doug Ravenel.

References

- 2001 Veblen Prize [1]

External links

- Michael J. Hopkins [2] at the Mathematics Genealogy Project
- Slides and video of lecture by Hopkins at Edinburgh, 21 April, 2009 [169]

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Witold Hurewicz

Not to be confused with Adolf Hurwitz.

Witold Hurewicz (June 29 1904 - September 6 1956) was a Polish mathematician.

Early life and education

He was born to a Jewish family in Łódź, Russian Empire (now Poland).

His father was an industrialist. Hurewicz attended school in a Russian controlled Poland but with World War I beginning before he had begun secondary school, major changes occurred in Poland. In August 1915 the Russian forces which had held Poland for many years withdrew. Germany and Austria-Hungary took control of most of the country and the University of Warsaw was refounded and it began operating as a Polish university. Rapidly, a strong school of mathematics grew up in the University of Warsaw, with → topology one of the main topics. Although Hurewicz knew intimately the topology that was being studied in Poland he chose to go to Vienna to continue his studies.

He studied under Hans Hahn and Karl Menger in Vienna, receiving a Ph.D. in 1926. Hurewicz was awarded a Rockefeller scholarship which allowed him to spend the year 1927-28 in Amsterdam. He was assistant to Brouwer in Amsterdam from 1928 to 1936. He was given study leave for a year which he decided to spend in the United States. He visited the Institute for Advanced Study in Princeton, New Jersey and then decided to remain in the United States and not return to his position in Amsterdam. Given the impending war in Europe this was clearly a wise decision.

Career

Hurewicz worked first at the University of North Carolina at Chapel Hill but during World War II he contributed to the war effort with research on applied mathematics. In particular, the work he did on servomechanisms at that time was classified because of its military importance. From 1945 until his death he worked at the Massachusetts Institute of Technology.

Hurewicz's early work was on set theory and → topology. The *Dictionary of Scientific Biography* describes it as: "...a remarkable result of this first period [1930] is his topological embedding of separable metric spaces into compact spaces of the same (finite) dimension.*"

In the field of general topology his contributions are centred around dimension theory. He wrote an important text with Henry Wallman, *Dimension Theory*, published in 1941. A reviewer writes that the book "...is truly a classic. It presents the theory of dimension for separable metric spaces with what seems to be an impossible mixture of depth, clarity, precision, succinctness, and comprehensiveness."

Hurewicz is best remembered for two remarkable contributions to mathematics, his discovery of the higher homotopy groups in 1935-36, and his discovery of exact sequences in 1941. His work led to → homological algebra. It was during Hurewicz's time as Brouwer's assistant in Amsterdam that he did the work on the higher homotopy groups; "...the idea was not new, but until Hurewicz nobody had pursued it as it should have been. Investigators did not expect much new information from groups, which were obviously commutative..."

Hurewicz had a second textbook published, but this was not until 1958 after his death. *Lectures on ordinary differential equations* is an introduction to ordinary differential equations which again reflects the clarity of his thinking and the quality of his writing.

He died during an outing at the International Symposium on Algebraic Topology in Uxmal, Mexico after tripping and falling off the top of a Mayan ziggurat. In the *Dictionary of Scientific Biography* it is suggested that he was "...a paragon of absentmindedness, a failing that probably led to his death."

See also

- Zygmunt Janiszewski

External links

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- Solomon Lefschetz *Witold Hurewicz, In memoriam* [3] Bull. Amer. Math. Soc. **63**, (1957), 77-82.

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[3] http://projecteuclid.org/euclid.bams/1183521492

Maxim Kontsevich

Maxim Kontsevich	
	
Born	25 August 1964 Russia
Citizenship	France
Fields	Mathematics
Institutions	Institut des Hautes Études Scientifiques University of Miami
Alma mater	Moscow State University, mexmat
Doctoral advisor	Don Bernard Zagier
Notable awards	Henri Poincaré Prize (1997) Fields Medal (1998) Crafoord Prize (2008)

Maxim Lvovich Kontsevich (Russian: Максим Львович Концевич) (born 25 August 1964) is a Russian mathematician. He received a Fields Medal in 1998, at the 23rd International Congress of Mathematicians in Berlin. He also received the Henri Poincaré Prize in 1997 and a Crafoord Prize in 2008.

Biography

Born into the family of Lev Rafailovich Kontsevich – Soviet orientalist and author of the Kontsevich system. After ranking second in the All-Union Mathematics Olympiads, he attended Moscow State University but left without a degree in 1985 to become a researcher at the Institute for Problems of Information Transmission in Moscow [1]. In 1992 he received his Ph.D. at the University of Bonn under Don Bernard Zagier. His thesis claims to prove a conjecture by Edward Witten that two quantum gravitational models are equivalent. Currently he is a Professor at the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette, France and Distinguished Professor at University of Miami in Coral Gables, Florida, U.S..

His work concentrates on geometric aspects of mathematical physics, most notably on knot theory, quantization, and mirror symmetry. His most famous result is a formal deformation quantization that holds for any Poisson manifold. He also introduced knot invariants defined by complicated integrals analogous to Feynman integrals. In topological field theory, he introduced the moduli space of stable maps, which may be considered a mathematically rigorous formulation of the Feynman integral for topological string theory. These results are a part of his "contributions to four problems of geometry" for which he was awarded the Fields Medal in 1998.

See also

- Kontsevich integral
- Homological mirror symmetry
- Motivic integration

References

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- Maxim Kontsevich [4] at the Mathematics Genealogy Project
- AMS Profile of Maxim Kontsevich [5]

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[2] http://www.icm2002.org.cn/general/prize/medal/1998.htm#Maxim
[3] http://www.icm2002.org.cn/
[4] http://genealogy.math.ndsu.nodak.edu/id.php?id=26861
[5] http://www.ams.org/featurecolumn/archive/kontsevich.html

Otto Hermann Künneth

Otto Hermann Lorenz Künneth (July 6, 1892 Neustadt an der Haardt – May 7, 1975 Erlangen) was a German mathematician and renowned → algebraic topologist, best known for his contribution to what is now known as the Künneth theorem.

His 1922 doctoral thesis was titled "Über die Bettischen Zahlen einer Produktmannigfaltigkeit" (University of Erlangen, supervised by Heinrich Franz Friedrich Tietze).

Künneth participated in the First World War and was captured by British forces.

Sources

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External links

- Otto Hermann Künneth [1] at the Mathematics Genealogy Project

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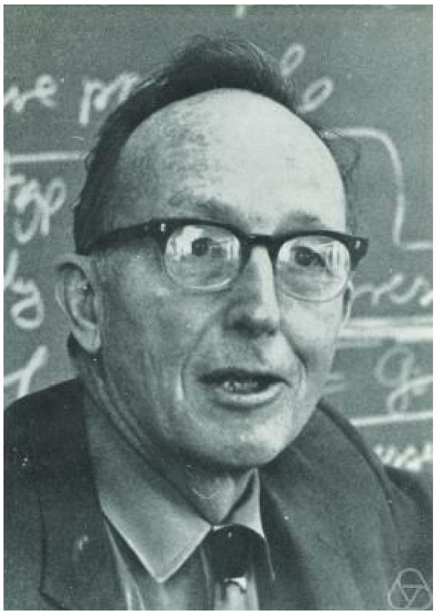
Saunders Mac Lane

Saunders Mac Lane (4 August 1909, Taftville, Connecticut – 14 April 2005, San Francisco) was an American mathematician who cofounded category theory with → Samuel Eilenberg.



Saunders Mac Lane

Career

Mac Lane was christened "Leslie Saunders MacLane", but "Leslie" fell into disuse because his parents, Donald MacLane and Winifred Saunders, came to dislike it. He began inserting a space into his surname because his first wife found it difficult to type the name without a space.[1]

Mac Lane earned a BA from Yale University in 1930, and an MA from the University of Chicago in 1931. During this period, he published his first scientific paper, in physics and co-authored with Irving Langmuir. He attended the University of Göttingen, 1931–1933, studying logic and mathematics under Paul Bernays, Emmy Noether and Hermann Weyl. Göttingen's Mathematisches Institut awarded him the Ph.D. in 1934.

From 1934 through 1938, Mac Lane held short term appointments at Harvard University, Cornell University, and the University of Chicago. He then held a tenure track appointment at Harvard, 1938–1947, before spending the rest of his career at the University of Chicago. In 1944 and 1945, he also directed Columbia University's Applied Mathematics Group, which was involved in the war effort as a contractor for the Applied Mathematics Panel.

Mac Lane served as vice president of the National Academy of Sciences and the American Philosophical Society, and as president of the American Mathematical Society. While presiding over the Mathematical Association of America in the 1950s, he initiated its activities aimed at improving the teaching of modern mathematics. He was a member of the National Science Board, 1974–1980, advising the American government. In 1976, he led a delegation of mathematicians to China to study the conditions affecting mathematics there. Mac Lane was elected to the National Academy of Sciences in 1949, and received the National Medal of Science in 1989.

Contributions

After a thesis in mathematical logic, his early work was in field theory and valuation theory. He wrote on valuation rings and Witt vectors, and separability in infinite field extensions. He started writing on group extensions in 1942, and began his epochal collaboration with → Samuel Eilenberg in 1943, resulting in what are now called Eilenberg–Mac Lane spaces $K(G,n)$, having a single non-trivial homotopy group G in dimension n. This work opened the way to group cohomology in general.

After introducing, via the Eilenberg-Steenrod axioms, the abstract approach to → homology theory, he and Eilenberg originated category theory in 1945. He is especially known for his work on coherence theorems. A recurring feature of category theory, abstract algebra, and of some other mathematics as well, is the use of diagrams, consisting of arrows (morphisms) linking objects, such as products and coproducts. According to McLarty (2005), this diagrammatic approach to contemporary mathematics largely stems from Mac Lane (1948).

Mac Lane had an exemplary devotion to writing approachable texts, starting with his very influential *A Survey of Modern Algebra*, coauthored in 1941 with Garrett Birkhoff. From then on, it was possible to teach elementary modern algebra to undergraduates using an English text. His *Categories for the Working Mathematician* remains the

definitive introduction to category theory.

Mac Lane supervised the Ph.Ds of, among many others, David Eisenbud, William Howard, Irving Kaplansky, Michael Morley, Anil Nerode, Robert Solovay, and John G. Thompson.

In addition to reviewing a fair bit of his mathematical output, the obituary articles McLarty (2005, 2007) clarify Mac Lane's contributions to the philosophy of mathematics. Mac Lane (1986) is an approachable introduction to his views on this subject.

Books by Mac Lane

- 1997 (1941). *A Survey of Modern Algebra* (with Garrett Birkhoff). A K Peters. ISBN 1-56881-068-7
- 1948, "Groups, categories and duality," *Proceedings of the Nat. Acad. of Sciences of the USA 34*: 263–67.
- 1995 (1963). *Homology*, Springer (Classics in Mathematics) ISBN 978-0387586625 (Originally, Band 114 of Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen.) AMS review. [2]
- 1999 (1967). *Algebra* (with Garrett Birkhoff). Chelsea. ISBN 0-8218-1646-2
- 1998 (1972). *Categories for the Working Mathematician*, Springer (Graduate Texts in Mathematics) ISBN 0-387-98403-8
- 1986. *Mathematics, Form and Function*. Springer-Verlag. ISBN 0-387-96217-4
- 1992. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (with Ieke Moerdijk). ISBN 0-387-97710-4
- 2005. *Saunders Mac Lane: A Mathematical Autobiography*. A K Peters. ISBN 1-56881-150-0

Biographical

- McLarty, Colin, 2005, "Saunders Mac Lane (1909–2005): His Mathematical Life and Philosophical Works, [3]" *Philosophia Mathematica 13*: 237-51. With selected bibliography emphasizing Mac Lane's philosophical writings.
- --------, 2007, "The Last Mathematician from Hilbert's Göttingen: Saunders Mac Lane as Philosopher of Mathematics, [4]", *British Journal for the Philosophy of Science 58(1)*: 77-112.

See also

- From Action to Mathematics per Mac Lane

External links

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- Obituary press release [6] from the University of Chicago.
- Photographs of Mac Lane, [7]1984–99.
- Kutateladze S.S., Saunders Mac Lane, the Knight of Mathematics [8]
- Saunders Mac Lane [9] at the Mathematics Genealogy Project

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[2] http://www.ams.org/bull/2000-37-01/S0273-0979-99-00847-2/S0273-0979-99-00847-2.pdf
[3] http://philmat.oxfordjournals.org/cgi/content/full/13/3/237
[4] http://bjps.oxfordjournals.org/cgi/content/short/58/1/77
[5] http://www-history.mcs.st-andrews.ac.uk/Biographies/MacLane.html
[6] http://www-news.uchicago.edu/releases/05/050421.maclane.shtml
[7] http://www.iti.cs.tu-bs.de/~koslowj/cgi-bin/p0454?MacLane-0/idx&256
[8] http://www.math.nsc.ru/LBRT/g2/english/ssk/knight_e.html
[9] http://genealogy.math.ndsu.nodak.edu/id.php?id=834

J.P.May

Jon Peter May (born September 16, 1939) is an American mathematician, working in the fields of → algebraic topology and category theory. He is one of the pioneers of abstract → homotopy theory, and invented operads and the May spectral sequence.

He received a B.A. from Swarthmore College in 1960, and a Ph.D. from Princeton University in 1964. His thesis, written under the direction of → John Moore, was titled *The cohomology of restricted Lie algebras and of Hopf algebras: Application to the Steenrod algebra*. From 1964 to 1967 he taught at Yale University. He has been a faculty member at the University of Chicago since 1967, and a Professor since 1970.

Research areas

Contributions

- May spectral sequence
- Operad theory

General areas

- → Algebraic topology
- Category theory
- → Homotopy theory

External links

- J.P. May [1] at the Mathematics Genealogy Project
- May's homepage at the University of Chicago [2]

References

[1] http://genealogy.math.ndsu.nodak.edu/id.php?id=6608
[2] http://www.math.uchicago.edu/~may/

John Coleman Moore

John Coleman Moore (b. ca. 1922 or 1923) is an American mathematician. He received his Ph.D. in 1952 from Brown University under the supervision of George W. Whitehead. His most heavily cited paper is on Hopf algebras, co-authored with John Milnor.[1] As a faculty member at Princeton University, he advised 23 students and is the academic ancestor of 582 mathematicians.[2] In 1983, a conference on → K-theory was held at Princeton in honor of his 60th birthday.[3] The Borel–Moore homology and Eilenberg–Moore spectral sequence are named after him.[4]

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[4] Rusin, Dave (1998), *People whose names are embedded in Math Subject Classification* (http://www.math.niu.edu/~rusin/known-math/98/MSC.names), . Updated February 2005.

Sergei Petrovich Novikov

Sergei Petrovich Novikov	
Born	20 March 1938 Gorky
Education	Moscow State University
Occupation	mathematician
Parents	Pyotr Sergeyevich Novikov, Ludmila Vsevolodovna Keldysh
Relatives	Mstislav Vsevolodovich Keldysh

Sergei Petrovich Novikov (also **Serguei**) (Russian: Сергей Петрович Новиков) (born 20 March 1938) is a Russian mathematician, noted for work in both → algebraic topology and soliton theory.

Early life

He was born in Gorky, Russian SFSR (now Nizhny Novgorod, Russia).

Novikov grew up in a family of talented mathematicians. His father was Pyotr Sergeyevich Novikov, who gave the negative solution of the word problem for groups. His mother Ludmila and uncle Mstislav were also important mathematicians.

In 1955 Novikov entered Moscow State University (graduating in 1960). Four years later he received the Moscow Mathematical Society Award for young mathematicians. In the same year he defended a dissertation for the *Candidate of Science in Physics and Mathematics* degree at the Moscow State University (it is equivalent to the PhD). In 1965 he defended a dissertation for the *Doctor of Science in Physics and Mathematics* degree there. In 1966 he became a Corresponding member of the USSR Academy of Sciences.

Research in topology

His early work was in cobordism theory, in relative isolation. Among other advances he showed how the Adams spectral sequence, a powerful tool for proceeding from → homology theory to the calculation of homotopy groups, could be adapted to the new (at that time) → cohomology theory typified by cobordism and → K-theory. This required the development of the idea of cohomology operations in the general setting, since the basis of the spectral sequence is the initial data of Ext functors taken with respect to a ring of such operations, generalising the Steenrod algebra. The resulting Adams-Novikov spectral sequence is now a basic tool in stable homotopy theory.

Novikov also carried out important research in geometric topology, being one of the pioneers with → William Browder, → Dennis Sullivan and Terry Wall of the surgery theory method for classifying high-dimensional manifolds. He proved the topological invariance of the rational Pontryagin classes, and posed the Novikov conjecture. This work was recognised by the award in 1970 of the Fields Medal. From about 1971 he moved to work in the field of isospectral flows, with connections to the theory of theta functions. Novikov's conjecture about the Riemann-Schottky problem (characterizing principally polarized abelian varieties that are the Jacobian of some algebraic curve) stated, essentially, that this was the case if and only if the corresponding theta function provided a solution to the Kadomtsev-Petviashvili equation of soliton theory. This was proved by T. Shiota in 1986. following earlier work by E. Arbarello and C. de Concini, and by M. Mulase.

Later career

Since 1971 Novikov has worked at the Landau Institute for Theoretical Physics of the USSR Academy of Sciences. In 1981 he was elected a Full Member of the USSR Academy of Sciences (Russian Academy of Sciences since 1991). In 1982 Novikov was also appointed the *Head of the Chair in Higher Geometry and Topology* at the Moscow State University.

In 1984 he was elected as a member of Serbian Academy of Sciences and Arts.

As of 2004, Novikov is the Head of the Department of geometry and topology at the Steklov Mathematical Institute. He also teaches at the University of Maryland, College Park and is a Principal Researcher of the Landau Institute for Theoretical Physics in Moscow.

In 2005 Novikov was awarded the Wolf Prize for his contributions to → algebraic topology, differential topology and to mathematical physics. He is one of just eleven mathematicians who received both the Fields Medal and the Wolf Prize.

Awards

- Lenin Prize (1967)
- Fields Medal (1970)
- Lobachevsky Medal (1981)
- Wolf Prize (2005)

External links

- Homepage [1] and Curriculum Vitae on the website of Steklov Mathematical Institute
- Biography (in Russian) [2] on the website of Moscow State University
- O'Connor, John J.; Robertson, Edmund F., "Sergei Petrovich Novikov [3]", *MacTutor History of Mathematics archive*.
- Sergei Petrovich Novikov [4] at the Mathematics Genealogy Project

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[1] http://www.mi.ras.ru/~snovikov/index.html
[2] http://higeom.math.msu.su/people/novikov/
[3] http://www-history.mcs.st-andrews.ac.uk/Biographies/Novikov_Sergi.html
[4] http://genealogy.math.ndsu.nodak.edu/id.php?id=55147

Daniel Quillen

Daniel Quillen	
Born	June 22, 1940
Nationality	American
Fields	Mathematics
Doctoral advisor	Raoul Bott
Notable awards	Fields Medal

Daniel Gray "Dan" Quillen (born June 22, 1940) is an American mathematician and a Fields Medalist.

From 1984 to 2006 he was the Waynflete Professor of Pure Mathematics at Magdalen College, Oxford. He is renowned for being the "prime architect" of higher → algebraic K-theory, for which he was awarded the Cole Prize in 1975 and the Fields Medal in 1978.

Education and career

Quillen was born in Orange, New Jersey. He entered Harvard University, where he earned both his BA (1961) and his PhD (1964), the latter of which was completed under the supervision of Raoul Bott with a thesis in partial differential equations. He was a Putnam Fellow in 1959.

Quillen obtained a position at the Massachusetts Institute of Technology after completing his doctorate. However, he also spent a number of years at several other universities. This experience would prove to be important in influencing the direction of his research. He visited France twice: first as a Sloan Fellow in Paris, during the academic year 1968–69, where he was greatly influenced by Grothendieck, and then, during 1973–74, as a Guggenheim Fellow. In 1969–70, he was a visiting member of the Institute for Advanced Study in Princeton, where he came under the influence of → Michael Atiyah.

In 1978, Quillen received a Fields Medal at the International Congress of Mathematicians held in Helsinki.

His Ph.D. students include Kenneth Brown, Howard Hiller, Jeanne Duflot, Mark Baker, Varghese Mathai (with whom he collaborated on the Mathai-Quillen formalism), and Jacek Brodzki.

Quillen retired at the end of 2006.

Mathematical contributions

Quillen's most celebrated contribution (mentioned specifically in his Fields medal citation) was his formulation of higher algebraic K-theory in 1972, a problem that had baffled mathematicians since algebraic K-theory was first formulated. This new tool, formulated in terms of homotopy theory, proved to be successful in formulating and solving major problems in algebra, particularly in ring theory and module theory. More generally, Quillen developed tools (especially his theory of model categories) which allowed algebro-topological tools to be applied in other contexts.

Before his ground-breaking work in defining higher algebraic K-theory, Quillen worked on the Adams conjecture, formulated by → Frank Adams in → homotopy theory. His proof of the conjecture used techniques from the modular representation theory of groups, which he later applied to work on cohomology of groups and → algebraic K-theory. He also worked on complex cobordism, showing that its formal group law is essentially the universal one.

In related work, he also supplied a proof of Serre's conjecture about the triviality of algebraic vector bundles on affine space.

He is also the architect (along with → Dennis Sullivan) of rational homotopy theory.

Selected publications

- Quillen, D. (1969), "Rational homotopy theory [1]", *Annals of Math* **90**: 205–295, doi:10.2307/1970725 [2], MR0258031 [3], http://links.jstor.org/sici?sici=0003-486X%28196909%292%3A90%3A2%3C205%3ARHT%3E2.0.CO%3B2-S

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[4] http://www-history.mcs.st-andrews.ac.uk/Biographies/Quillen.html
[5] http://genealogy.math.ndsu.nodak.edu/id.php?id=13325

Jean-Pierre Serre

Jean-Pierre Serre	
	
Born	15 September 1926 Bages, Pyrénées-Orientales, France
Residence	Paris, France
Nationality	France
Fields	Mathematics
Institutions	Centre National de la Recherche Scientifique Collège de France
Alma mater	École Normale Supérieure
Doctoral advisor	→ Henri Cartan
Doctoral students	Michel Broué John Labute
Notable awards	Abel Prize (2003) Fields Medal (1954) Wolf Prize in Mathematics (2000) Balzan Prize (1985)

Jean-Pierre Serre (born 15 September 1926) is a French mathematician in the fields of algebraic geometry, number theory and → topology. He has received numerous awards and honors for his mathematical research and exposition, including the Fields Medal in 1954 and the Abel Prize in 2003.

Biography

Early years

Born in Bages, Pyrénées-Orientales, France, Serre was educated at the Lycée de Nîmes and then from 1945 to 1948 at the École Normale Supérieure in Paris. He was awarded his doctorate from the Sorbonne in 1951. From 1948 to 1954 he held positions at the Centre National de la Recherche Scientifique in Paris. In 1956 he was elected professor at the Collège de France, a position he held until his retirement in 1994.

Career

From a very young age he was an outstanding figure in the school of → Henri Cartan, working on → algebraic topology, several complex variables and then commutative algebra and algebraic geometry, in the context of sheaf theory and → homological algebra techniques. Serre's thesis concerned the Leray–Serre spectral sequence associated to a fibration. Together with Cartan, Serre established the technique of using Eilenberg-MacLane spaces for computing homotopy groups of spheres, which at that time was considered as the major problem in topology.

In his speech at the Fields Medal award ceremony in 1954, Hermann Weyl praised Serre in seemingly extravagant terms, and also made the point that the award was for the first time awarded to an algebraist. Serre subsequently changed his research focus; he apparently thought that → homotopy theory, where he had started, was already overly technical. However, Weyl's perception that the central place of classical analysis had been challenged by abstract algebra has subsequently been justified, as has his assessment of Serre's place in this change.

Algebraic geometry

In the 1950s and 1960s, a fruitful collaboration between Serre and the two-years-younger → Alexander Grothendieck led to important foundational work, much of it motivated by the Weil conjectures. Two major foundational papers by Serre were *Faisceaux Algébriques Cohérents* (FAC), on coherent cohomology) and *Géometrie Algébrique et Géométrie Analytique* (GAGA).

Even at an early stage in his work Serre had perceived a need to construct more general and refined cohomology theories to tackle the Weil conjectures. The problem was that the cohomology of a coherent sheaf over a finite field couldn't capture as much topology as singular cohomology with integer coefficients. Amongst Serre's early candidate theories of 1954–55 was one based on Witt vector coefficients.

Around 1958 Serre suggested that isotrivial principal bundles on algebraic varieties — those that become trivial after pullback by a finite étale map — are important. This acted as one important source of inspiration for Grothendieck to develop étale topology and the corresponding theory of étale cohomology. These tools, developed in full by Grothendieck and collaborators in Séminaire de géométrie algébrique (SGA) 4 and SGA 5, provided the tools for the eventual proof of the Weil conjectures.

In later years Serre was sometimes a source of counterexamples to over-optimistic extrapolations. He also had a close working relationship with Pierre Deligne, who eventually finished the proof of the Weil conjectures.

Other work

From 1959 onward Serre's interests turned towards number theory, in particular class field theory and the theory of complex multiplication.

Amongst his most original contributions were: the concept of → algebraic K-theory; the Galois representation theory for ℓ-adic cohomology and the conceptions that these representations were "large"; and the Serre conjecture on mod-p representations that made Fermat's last theorem a connected part of mainstream arithmetic geometry.

Honours and awards

Serre, at twenty-seven in 1954, is the youngest ever to be awarded the Fields Medal. In 1985, he went on to win the Balzan Prize, the Steele Prize in 1995, the Wolf Prize in Mathematics in 2000, and was the first recipient of the Abel Prize in 2003. Serre and John Thompson are the only laureates of all three of the Fields Medal, the Wolf Prize, and the Abel Prize.

See also

- Serre duality
- Serre's multiplicity conjectures
- Serre's property FA
- → Serre spectral sequence
- Serre fibration
- Serre twist sheaf
- Thin set in the sense of Serre
- Quillen-Suslin theorem

- → Nicolas Bourbaki

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Dennis Sullivan

Dennis Sullivan

Dennis Parnell Sullivan (born 1941, Port Huron, Michigan) is an American mathematician. He is known for work in topology, both algebraic and geometric, and on dynamical systems. He holds the Albert Einstein Chair at the City University of New York Graduate Center, and is a professor at Stony Brook University.

Work

His 1966 doctorate was from Princeton University. His thesis, entitled *Triangulating homotopy equivalences*, was written under the supervision of → William Browder, and was a major contribution to surgery theory. He was a permanent member of the Institut des Hautes Études Scientifiques from 1974 to 1997.

Sullivan is one of the founders of the surgery method of classifying high-dimensional manifolds, along with → Browder, Sergei Novikov and C. T. C. Wall. In → homotopy theory, Sullivan put forward the radical concept that spaces could directly be *localised*, a procedure hitherto applied to the algebraic constructs made from them. He founded (along with → Daniel Quillen) rational homotopy theory.

The Sullivan conjecture, proved in its original form by Haynes Miller, states that the classifying space *BG* of a finite group *G* is sufficiently different from any finite CW complex *X*, that it maps to such an *X* only 'with difficulty'; in a more formal statement, the space of all mappings *BG* to *X*, as pointed spaces and given the compact-open topology, is weakly contractible. This area has generated considerable further research. (Both these matters are discussed in his 1970 MIT notes [1].)

In 1985, he proved the No wandering domain theorem. The Parry-Sullivan invariant is named after him and the English mathematician Bill Parry.

In 1987, he proved Thurston's conjecture about the approximation of the Riemann map by circle packings together with Burton Rodin.

Awards and honors

Awards include the 1971 Oswald Veblen Prize in Geometry, the 1981 Prix Élie Cartan of the French Academy of Sciences, the King Faisal International Prize for Science in 1994, the 2004 National Medal of Science and the 2006 AMS Steele Prize.

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René Thom

René Thom	
 René Thom in Nice, 1970	
Born	September 2, 1923 Montbéliard, France
Died	October 25, 2002 (aged 79) Bures-sur-Yvette, France
Fields	Mathematics
Alma mater	University of Paris
Doctoral advisor	→ Henri Cartan
Doctoral students	David Trotman
Known for	→ topology
Notable awards	Fields Medal in 1958

René Frédéric Thom (September 2, 1923 – October 25, 2002) was a French mathematician. He made his reputation as a topologist, moving on to aspects of what would be called singularity theory; he became world-famous among the wider academic community and the educated general public for one aspect of this latter interest, his work as founder of catastrophe theory (later developed by Christopher Zeeman). He received the Fields Medal in 1958.

Biography

René Thom was born in Montbéliard, France. He was educated at the Lycée Saint-Louis and the École Normale Supérieure, both in Paris. He received his PhD in 1951 from the University of Paris. His thesis, titled *Espaces fibrés en sphères et carrés de Steenrod* (*Sphere bundles and Steenrod squares*), was written under the direction of → Henri Cartan. The foundations of cobordism theory, for which he received the Fields Medal at Edinburgh in 1958, were already present in his thesis.

After a fellowship in the United States, he went on to teach at the Universities of Grenoble (1953-1954) and Strasbourg (1954-1963), where he was appointed Professor in 1957. In 1964, he moved to the Institut des Hautes Études Scientifiques, in Bures-sur-Yvette. He was awarded the Grand Prix Scientifique de la Ville de Paris in 1974, and became a Member of the Academie des Sciences of Paris in 1976.

While René Thom is most known to the public for his development of catastrophe theory between 1968 and 1972, his earlier work was on differential topology. In the early 1950s it concerned what are now called Thom spaces, characteristic classes, cobordism theory, and the Thom transversality theorem. Another example of this line of work is the Thom conjecture, versions of which have been investigated using gauge theory. From the mid 50's he moved into singularity theory, of which catastrophe theory is just one aspect, and in a series of deep (and at the time

obscure) papers between 1960 and 1969 developed the theory of stratified sets and stratified maps, proving a basic stratified isotopy theorem describing the local conical structure of Whitney stratified sets, now known as the Thom-Mather isotopy theorem. Much of his work on stratified sets was developed so as to understand the notion of topologically stable maps, and to eventually prove the result that the set of topologically stable mappings between two smooth manifolds is a dense set. Thom's lectures on the stability of differentiable mappings, given at Bonn in 1960, were written up by Harold Levine and published in the proceedings of a year long symposium on singularities at Liverpool University during 1969-70, edited by Terry Wall. The proof of the density of topologically stable mappings was completed by John Mather in 1970, based on the ideas developed by Thom in the previous ten years. A coherent detailed account was published in 1976 by C. Gibson, K. Wirthmuller, E. Looijenga and A. du Plessis.

During the last twenty years of his life Thom's published work was mainly in philosophy and epistemology, and he undertook a reevaluation of Aristotle's writings on science.

Beyond Thom's contributions in algebraic topology, his influence on modern differential geometry, through the intensive study of generic properties, can hardly be exaggerated.

Thom died on October 25, 2002, in Bures-sur-Yvette.

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Hassler Whitney

Hassler Whitney	
Born	March 23, 1907
Died	May 10, 1989 (aged 82)
Fields	Mathematics
Institutions	Harvard University Institute for Advanced Study Princeton University National Science Foundation National Defense Research Committee
Alma mater	Yale University
Doctoral advisor	George David Birkhoff
Known for	→ algebraic topology differential topology singularity theory
Notable awards	National Medal of Science 1976 Wolf Prize 1983 Steele Prize 1985

Hassler Whitney (23 March 1907 – 10 May 1989) was an American mathematician. He was one of the founders of singularity theory, and did foundational work in manifolds, embeddings, immersions, and characteristic classes.

Work

Whitney's earliest work, from 1930 to 1933, was on graph theory. Many of his contributions were to the graph-coloring, and the ultimate computer-assisted solution to the four-color problem relied on some of his results. His work in graph theory culminated in a 1935 paper, where he laid the foundations for matroids, a fundamental notion in modern combinatorics and representation theory.

Whitney's lifelong interest in geometric properties of functions also began around this time. His earliest work in this subject was on the possibility of extending a function defined on a closed subset of $\mathbf{R}^n$ to a function on all of $\mathbf{R}^n$ with certain smoothness properties. A complete solution to this problem was only found in 2005 by Charles Fefferman.

In a 1936 paper, Whitney gave a definition of a smooth manifold of class C^r, and proved that, for high enough values of r, a smooth manifold of dimension n may be embedded in $\mathbf{R}^{2n+1}$, and immersed in $\mathbf{R}^{2n}$. (In 1944 he managed to reduce the dimension of the ambient space by 1, so long as $n > 2$, by a technique that has come to be known as the "Whitney trick.") This basic result shows that manifolds may be treated intrinsically or extrinsically, as we wish. The intrinsic definition had only been published a few years earlier in the work of Oswald Veblen and J.H.C. Whitehead. These theorems opened the way for much more refined studies: of embedding, immersion and also of smoothing, that is, the possibility of having various smooth structures on a given topological manifold.

He was one of the major developers of → cohomology theory, and characteristic classes, as these concepts emerged in the late 1930s, and his work on algebaic topology continued into the 40s. He also returned to the study of functions in the 1940s, continuing his work on the extension problems formulated a decade earlier, and answering a question of Schwarz in a 1948 paper *On Ideals of Differentiable Functions*.

Whitney had, throughout the 1950s, an almost unique interest in the topology of singular spaces and in singularities of smooth maps. An old idea, even implicit in the notion of a simplicial complex, was to study a singular space by decomposing it into smooth pieces (nowadays called "strata"). Whitney was the first to see any subtlety in this

definition, and pointed out that a good "stratification" should satisfy conditions he termed "A" and "B". The work of → René Thom and John Mather in the 1960s showed that these conditions give a very robust definition of stratified space. The singularities in low dimension of smooth mappings, later to come to prominence in the work of → René Thom, were also first studied by Whitney.

His book *Geometric Integration Theory* gives a theoretical basis for Stokes' theorem applied with singularities on the boundary and later inspired the generalization found by Jenny Harrison.

These aspects of Whitney’s work have looked more unified, in retrospect and with the general development of singularity theory. Whitney’s purely topological work (Stiefel–Whitney class, basic results on vector bundles) entered the mainstream more quickly.

Career

He received his Ph.D. from Yale University in 1928; his Mus.B., 1929; Sc.D. (Honorary), 1947; and Ph.D. from Harvard University, under George David Birkhoff, in 1932.

He was Instructor of Mathematics at Harvard University, 1930-31, 1933-35; NRC Fellow, Mathematics, 1931-33; Assistant Professor, 1935-40; Associate Professor, 1940-46, Professor, 1946-52; Professor Instructor, Institute for Advanced Study, Princeton University, 1952-77; Professor Emeritus, 1977-89; Chairman of the Mathematics Panel, National Science Foundation, 1953-56; Exchange Professor, College de France, 1957; Memorial Committee, Support of Research in Mathematical Sciences, National Research Council, 1966-67; President, International Commission of Mathematical Instruction, 1979-82; Research Mathematicians, National Defense Research Committee, 1943-45; Construction of the School of Mathematics. Recipient, National Medal of Science, 1976, Wolf Prize, Wolf Foundation, 1983; and a Steele Prize in 1985.

He was a member of the National Academy of Science; Colloquium Lecturer, American Mathematical Society, 1946; Vice President, 1948-50 and Editor, American Journal of Mathematics, 1944-49; Editor, Mathematical Reviews, 1949-54; Chairman of the Committee vis. lectureship, 1946-51; Committee Summer Instructor, 1953-54; Steele Prize, 1985, American Mathematical Society; American National Council Teachers of Mathematics, London Mathematical Society (Honorary), Swiss Mathematics Society (Honorary), Académie des Sciences de Paris (Foreign Associate); New York Academy of Sciences.

Family

Hassler Whitney was the son of First District New York Supreme Court judge Edward Baldwin Whitney and Josepha (Newcomb) Whitney, the grandson of Yale University Professor of Ancient Languages William Dwight Whitney, the great-grandson of Connecticut Governor and US Senator Roger Sherman Baldwin, and the great-great-great-grandson of American founding father Roger Sherman.

Hassler Whitney's maternal grandparents were professor & astronomer Simon Newcomb and Mary Hassler Newcomb (the granddaughter of the first superintendent of the Coast Survey - Ferdinand Hassler).

Married Margaret R. Howell, May 30, 1930; children: James Newcomb, Carol, Marian; married Mary Barnett Garfield, January 16, 1955; children: Sarah Newcomb, Emily Baldwin; and married Barbara Floyd Osterman, February 8, 1986.

See also

Named after Whitney

- Loomis–Whitney inequality
- McShane–Whitney extension theorem
- Stiefel–Whitney class
- Whitney's conditions A and B
- Whitney embedding theorem
- Whitney graph isomorphism theorem
- Whitney immersion theorem
- Whitney trick
- Whitney umbrella

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J.H.C.Whitehead

J. H. C. Whitehead	
 John Henry Constantine Whitehead	
Born	11 November 1904 Madras (Chennai), India
Died	8 May 1960 (aged 55) Princeton, New Jersey
Residence	United Kingdom, U.S.
Nationality	British
Fields	Mathematics
Institutions	Oxford University
Alma mater	Oxford University Princeton University
Doctoral advisor	Oswald Veblen
Doctoral students	Michael Barratt Ronald M. Brown Wilfred H. Cockroft Victor K. A. M. Gugenheim Graham Higman Peter Hilton Ioan James Brian Steer
Known for	CW complex Simple homotopy Whitehead group Whitehead manifold Whitehead product

John Henry Constantine Whitehead (11 November 1904–8 May 1960), known as Henry, was a British mathematician and was one of the founders of → homotopy theory. He was born in Chennai (then known as Madras), in India, and died in Princeton, New Jersey, in 1960.

Life

J. H. C. (Henry) Whitehead was the son of the Right Rev. Henry Whitehead, Bishop of Madras and brother of A. N. Whitehead, and of Isobel Duncan, who had herself studied mathematics at Oxford. He was brought up in Oxford, went to Eton and read mathematics at Balliol College, Oxford. After a year working as a stockbroker, he started a Ph.D. in 1929 at Princeton University. His thesis, titled *The representation of projective spaces*, was written under the direction of Oswald Veblen in 1930. While in Princeton, he also worked with Solomon Lefschetz.

He became a fellow of Balliol in 1933. In 1934 he married the concert pianist Barbara Smyth, great-great-granddaughter of Elizabeth Fry and a cousin of Peter Pears; they had two sons. During the Second World War he worked on operations research for submarine warfare. Later, he joined the codebreakers at Bletchley Park, and by 1945 was one of some fifteen mathematicians working in the "Newmanry", a section headed by Max Newman and responsible for breaking a German teleprinter cipher using machine methods.[1] Those methods included the Colossus machines, early digital electronic computers.[1]

From 1947 to 1960 he was the Waynflete Professor of Pure Mathematics at Magdalen College, Oxford.

He became president of the London Mathematical Society (LMS) in 1953, a post he held until 1955.[2] The LMS established two prizes in memory of J. H. C. Whitehead. The first is the annually awarded, to multiple recipients, Whitehead Prize; the second a biennially awarded Senior Whitehead Prize.[3]

In the late 1950s, Whitehead approached Robert Maxwell, then chairman of Pergamon Press, to start a new journal, *Topology*, but died before its first edition appeared in 1962.

Work

His definition of CW complexes gave a setting for homotopy theory that became standard. He introduced the idea of → simple homotopy theory, which was later much developed in connection with → algebraic K-theory. The Whitehead product is an operation in homotopy theory. The Whitehead problem on abelian groups was solved (as an independence proof) by Saharon Shelah. His involvement with topology and the Poincaré conjecture led to the creation of the Whitehead manifold. The definition of crossed modules is due to him. Whitehead also made important contributions in differential topology, particularly on triangulations and their associated smooth structures.

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See also

- Simple homotopy
- Spanier-Whitehead duality
- Whitehead group
- Whitehead link
- Whitehead theorem
- Whitehead torsion

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Important Theorems in Algebraic Topology

Borsuk-Ulam theorem

In mathematics, the **Borsuk–Ulam theorem** states that any continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point. (Two points on a sphere are called antipodal if they are in exactly opposite directions from the sphere's center.)

The case $n = 2$ is often illustrated by saying that at any moment there is always a pair of antipodal points on the Earth's surface with equal temperatures and equal barometric pressures. This assumes that temperature and barometric pressure vary continuously.

The Borsuk–Ulam theorem was first conjectured by Stanisław Ulam. It was proved by → Karol Borsuk in 1933.

There is an elementary proof that the Borsuk–Ulam theorem implies the → Brouwer fixed point theorem.

A stronger statement related to Borsuk–Ulam theorem is that every antipode-preserving map

$$f : \mathbb{S}^n \to \mathbb{S}^n$$

has odd degree.

Corollaries of Borsuk-Ulam theorem

- No subset of $\mathbf{R}^n$ is homeomorphic to $\mathbf{S}^n$.
- The Lusternik-Schnirelmann theorem: If the sphere $\mathbf{S}^n$ is covered by $n + 1$ open sets, then one of these sets contains a pair $(x, -x)$ of antipodal points. (this is equivalent to the Borsuk-Ulam theorem)
- The Ham sandwich theorem: For any compact sets $A_1, \ldots, A_n$ in $\mathbf{R}^n$ we can always find a hyperplane dividing each of them into two subsets of equal measure).

See also

- Sperner's lemma
- Tucker's lemma
- Topological combinatorics
- Necklace splitting problem
- Kakutani's theorem (geometry)
- Isovariant

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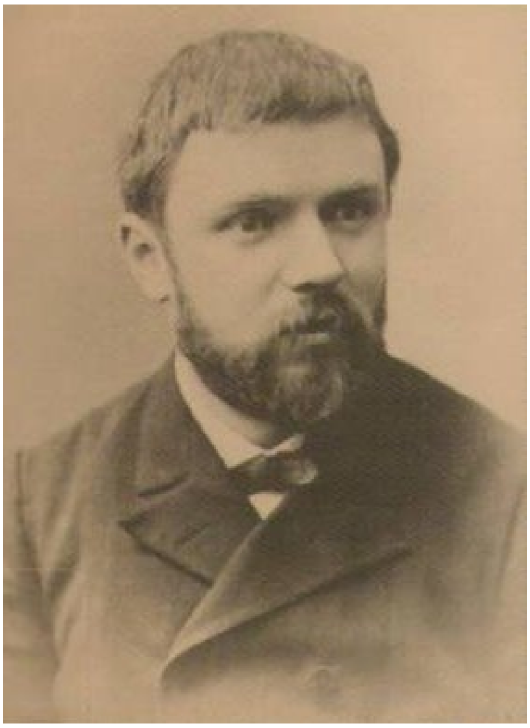
Brouwer fixed point theorem



In 1886, Henri Poincaré proved a result that is equivalent to Brouwer's fixed point theorem. The three-dimensional case of the exact statement was proved in 1904 by Piers Bohl, and the general case in 1910 by Jacques Hadamard. → Luitzen Egbertus Jan Brouwer proposed a new proof in 1912.

In mathematics, **Brouwer's fixed point theorem** is a theorem in → topology, named after → Luitzen Brouwer. It is one of many fixed point theorems, which state that for any continuous function f with certain properties there is a point x_0 such that $f(x_0) = x_0$. The simplest form of Brouwer's theorem is for continuous functions f from a disk D to itself. A more general form is for continuous functions from a convex compact subset K of Euclidean space to itself.

Among hundreds of fixed point theorems[1] , Brouwer's is particularly well known, due in part to the fact that it is used across numerous fields of mathematics. In its original field, this result is one of the key theorems characterizing the topology of Euclidean spaces, along with the Jordan curve theorem, the hairy ball theorem or the Borsuk–Ulam theorem.[2] This gives it a place among the fundamental theorems of topology[3] . The theorem is also used for proving deep results about differential equations and is covered in most introductory courses on differential geometry. It appears in unlikely fields such as game theory, where John Nash used it to prove the existence of a winning strategy for the game Hex.

The theorem was first studied in view of work on differential equations by the French mathematicians around Poincaré and Picard. Proving results such as the Poincaré–Bendixson theorem requires the use of topological methods. This work at the end of the 19th century opened into several successive versions of the theorem. The general case was first proved in 1910 by Hadamard, and then in 1912 by → Luitzen Egbertus Jan Brouwer.

Statement

The theorem has several formulations, depending on the context in which it is used. The simplest is sometimes given as follows:

In the plane

Every continuous function f from a closed disk to itself has at least one fixed point.[4] .

This can be generalized to an arbitrary finite dimension:

In Euclidean space

Every continuous function from a closed ball of a Euclidean space to itself has a fixed point.[5]

A slightly more general version is as follows:[6]

Convex compact set

Every continuous function f from a convex compact subset K of a Euclidean space to K itself has a fixed point.[7]

An even more general form is better known under a different name:

Schauder fixed point theorem

Every continuous function from a convex compact subset K of a Banach space to K itself has a fixed point.[8]

Notes

The function f in this theorem is not required to be bijective or even surjective. Since any closed ball in Euclidean n-space is homeomorphic to the closed unit ball D^n, the theorem also has equivalent formulations that only state it for D^n.

Because the properties involved (continuity, being a fixed point) are invariant under homeomorphisms, the theorem is equivalent to forms in which the domain is required to be a closed unit ball D^n. For the same reason it holds for every set that is homeomorphic to a closed ball (and therefore also closed, bounded, connected, without holes, etc.).

The statement of the theorem is false if formulated for the *open* unit disk, the set of points with distance strictly less than 1 from the origin. Consider for example the function

$$f(x,y) = (\tfrac{1}{2}(x+\sqrt{1-y^2}), y)$$

which maps every point of the open unit disk in $\mathbf{R}^2$ to another point of the open unit disk slightly to the right of the given one.

Illustrations

The theorem has several "real world" illustrations. For example: take two sheets of graph paper of equal size with coordinate systems on them, lay one flat on the table and crumple up (without ripping or tearing) the other one and place it any fashion on top of the first so that the crumpled paper does not reach outside the flat one. There will then be at least one point of the crumpled sheet that lies directly above its corresponding point (i.e. the point with the same coordinates) of the flat sheet. This is a consequence of the $n = 2$ case of Brouwer's theorem applied to the continuous map that assigns to the coordinates of every point of the crumpled sheet the coordinates of the point of the flat sheet immediately beneath it.

In three dimensions the consequence of the Brouwer fixed point theorem is that no matter how much you stir or shake a cocktail in a glass some point in the liquid will remain in the exact same place in the glass as before you took any action, assuming that the final position of each point is a continuous function of its original position, and that the liquid after stirring or shaking is contained within the space originally taken up by it.

Another consequence of the case $n = 3$ is given by an informational display of a map in, for example, an airport terminal. The function that sends points of the terminal to their image on the map is continuous and therefore has a fixed point, usually indicated by a cross or arrow with the text *You are here*. A similar display outside the terminal would violate the condition that the function is "to itself" and fail to have a fixed point. For this example, the existence of a fixed point is also a consequence of the Banach fixed point theorem, since the function mapping points in space to the display is a contraction mapping.

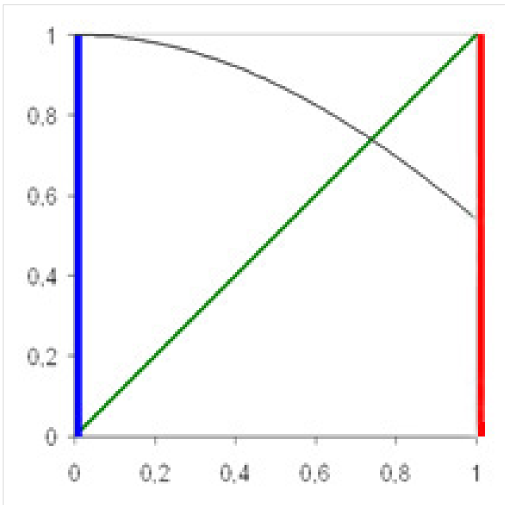
Intuitive approach

Explanations attributed to Brouwer

The theorem is supposed to have originated from Brouwer's observation of a cup of coffee.[9] If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving.[10] The fixed point is not necessarily the point that seems to be motionless, since the centre of the turbulence moves a little bit. The result is not intuitive, since the original fixed point may become mobile when another fixed point appears.

Brouwer is said to have added: "I can formulate this splendid result different, I take a horizontal sheet, and another identical one which I crumple, flatten and place on the other. Then a point of the crumpled sheet is in the same place as on the other sheet."[10] Brouwer "flattens" his sheet as with a flat iron, without removing the folds and wrinkles.

One-dimensional case



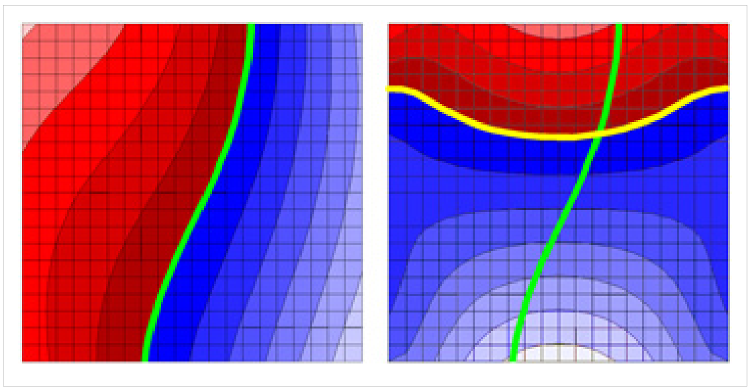
In one dimension, the result is intuitive and easy to prove. The continuous function f is defined on a closed interval $[a, b]$ and takes values in the same interval. Saying that this function has a fixed point amounts to saying that its graph (black in the figure on the right) intersects that of the function defined on the same interval $[a, b]$ which maps x to x (green).

Intuitively, any continuous line from the left edge of the square to the right edge must necessarily intersect the green diagonal.

It is not hard to give a formal proof. It suffices to consider the function g which maps x to $f(x) - x$. It is ≥ 0 on a and ≤ 0 on b. By the intermediate value theorem, g has a zero in $[a, b]$; this zero is a fixed point.

Brouwer is said to have expressed this as follows: "Instead of examining a surface, we will prove the theorem about a piece of string. Let us begin with the string in an unfolded state, then refold it. Let us flatten the refolded string. Again a point of the string has not changed its position with respect to its original position on the unfolded string."[10]

Two-dimensional case



In the plane, an intuitive argument shows that the result is *probably* true. Nevertheless the proof is tricky. If K, the domain of f, has empty interior, then it is a line segment. Otherwise K is homeomorphic to a closed unit ball, i.e. there is a homeomorphism $\varphi: D^2 \to K$ from the closed unit ball to K. The equation defining the fixed point can therefore be written as $h(x) = x$, where h denotes the composed function $\varphi^{-1} \circ f \circ \varphi$.

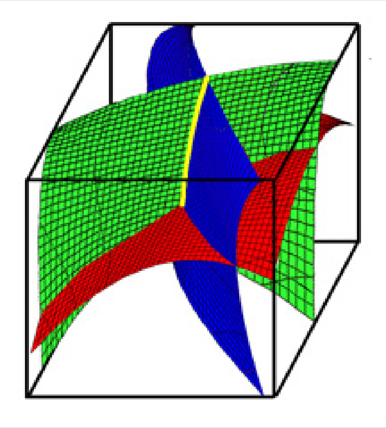
In other words, one can assume that K is a closed unit ball. The norm can moreover be chosen arbitrarily. Choosing the maximum norm (the maximum of the absolute values of the coordinates) shows that without loss of generality K can be assumed to be the set [-1, 1]×[-1, 1].

Denoting by g the function that maps x to $h(x) - x$, the goal is to prove that the zero vector is in the image of g on [-1, 1]×[-1, 1]. If g_k, for k equal to 1 or 2, are the two coordinate functions of g, this amounts to proving the existence of a point x_0 such that g_1 and g_2 both have a zero at x_0.

The function g_1 goes from [-1, 1]×[-1, 1] to [-1, 1]. It can be interpreted as a map of a region which indicates the altitude of every point (see first figure on the right). In the area {-1}×[-1, 1], this altitude is ≥0 (red in the figure), while on {1}×[-1, 1] it is ≤0 (blue). This suggests that the contour line 0 is a line (green in the figure) from a point in [-1, 1]×{1} to a point in [-1, 1]×{-1}. The same reasoning applied to g_2 suggests that the contour line 0 is this time a line from somewhere in {-1}×[-1, 1] to somewhere in {1}×[-1, 1] (see yellow line in the second figure).

Intuitively, it seems obvious that these two contour lines (green and yellow) must necessarily intersect, and the point of intersection is a fixed point of foφ. Verifying this intuition is not as easy as it appears. The green zone is not necessarily a connected line, not even necessarily a line. In fact, it does not even have to contain a line!

Finite-dimensional case



The intuitive approach of the previous section generalizes to any finite dimension. To understand how, it is sufficient to look at dimension 3.

The goal is again to prove that the function g, which now has three coordinates, has a zero. The first coordinate is ≥ 0 on the left face of the cube and ≤ 0 on the right face. There is every reason to think that the set of zeros contains a sheet, shown in blue in the figure on the right. This sheet *cuts* the cube into two connected components, one containing part of the right face and one containing part of the left face.

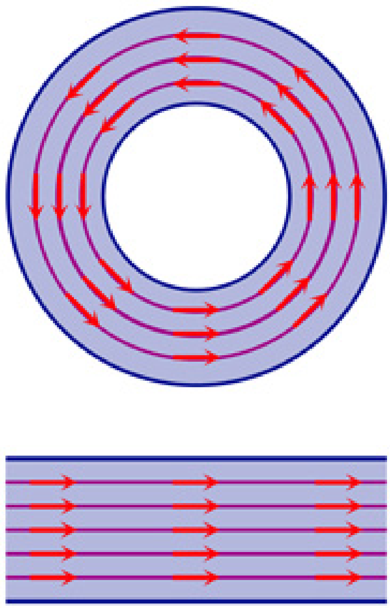
Assuming that the y axis is in *front-back* direction, the same reasoning suggests the existence of a sheet, shown in green in the figure, which also *cuts* the cube into at least two connected components. The intersection of the two sheets probably contains a *line*, depicted in yellow, going from the upper face to the lower face.

Now the third component of g describes a face shown in red. It appears that this sheet must intersect the yellow line. The point of intersection is the desired fixed point.

History

The Brouwer fixed point theorem was one of the early achievements of → algebraic topology, and is the basis of more general fixed point theorems which are important in functional analysis. The case $n = 3$ first was proved by Piers Bohl in 1904 (published in *Journal für die reine und angewandte Mathematik*). It was later proved by → L. E. J. Brouwer in 1909. Jacques Hadamard proved the general case in 1910, and Brouwer found a different proof in 1912. Since these early proofs were all non-constructive indirect proofs, they ran contrary to Brouwer's intuitionist ideals. Methods to construct (approximations to) fixed points guaranteed by Brouwer's theorem are now known, however; see for example (Karamadian 1977) and (Istrăţescu 1981).

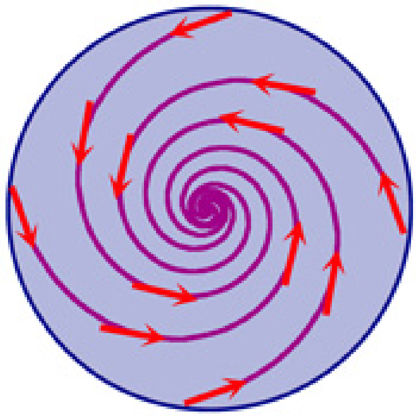
Prehistory



For flows in an unbounded area, or in an area with a "hole", the theorem is not applicable.

To understand the prehistory of Brouwer's fixed point theorem one needs to pass through differential equations. At the end of the 19th century, the old problem[11] of the stability of the solar system returned into the focus of the mathematical community.[12] Its solution required new methods. As noted by Henri Poincaré, who worked on the three-body problem, there is no hope to find an exact solution: "Nothing is more proper to give us an idea of the hardness of the three-body problem, and generally of all problems of Dynamics where there is no uniform integral and the Bohlin series diverge."[13] He also noted that the search for an approximate solution is no more efficient: "the more we seek to obtain precise approximations, the more the result will diverge towards an increasing imprecision."[14] .

He studied a question analogous to that of the surface movement in cup of coffee. What can we say, in general, about the trajectories on a surface animated by a constant flow?[15] Poincaré discovered that the answer can be found in what we now call the → topological properties in the area containing the trajectory. If this area is compact, i.e. bothclosed and bounded, then the trajectory either becomes stationary, or it approaches a limit cycle.[16] Poincaré went further; if the area is of the same kind as a disk, as is the case for the cup of coffee, there must necessarily be a fixed point. This fixed point is invariant under all functions which associate to each point of the original surface its position after a short time interval *t*. If the area is a circular band, or if it is not closed[17] , then this is not necessarily the case.



The theorem applies to any disk-shaped area, where it guarantees the existence of a fixed point.

To understand differential equations better, a new branch of mathematics was born. Poincaré called it *analysis situs*. The French Encyclopædia Universalis defines it as the branch which "treats the properties of an object that are invariant if it is deformed in any continuous way, without tearing"[18] . In 1886, Poincaré proved a result that is equivalent to Brouwer's fixed point theorem[19] , although the connection with the subject of this article was not yet apparent[20] . A little later, he developed one of the fundamental tools for better understanding the analysis situs, now known as the → fundamental group or sometimes the Poincaré group.[21] . This method can be used for a very compact proof of the theorem under discussion.

Poincaré's method was analogous to that of Emile Picard, a contemporary mathematician who generalized the Cauchy–Lipschitz theorem.[22] . Picard's approach is based on a result that would later be formalised by another fixed point theorem, named after Banach. Instead of the topological properties of the domain, this theorem uses the fact that the function in question is a contraction.

First proofs

Hadamard played the role of a midwife, helping Brouwer to formalize his ideas.

At the dawn of the 20th century, the interest in analysis situs did not stay unnoticed. However, the necessity of a theorem equivalent to the one discussed in this article was not yet evident. Piers Bohl, a Latvian mathematician, applied topological methods to the study of differential equations.[23] . In 1904 he proved the three-dimensional case of our theorem, but his publication was not noticed.[24] .

It was Brouwer, finally, who gave the theorem its first patent of nobility. His goals were different from those of Poincaré. This mathematician was inspired by the foundations of mathematics, especially mathematical logic and → topology. His initial interest lay in an attempt to solve Hilberts fifth problem.[25] In 1909, during a voyage to Paris, he met Poincaré, Hadamard and Borel. The ensuing discussions convinced Brouwer of the importance of a better understanding of Euclidean spaces, and were the origin of a fruitful exchange of letters with Hadamard. For the next four years, he concentrated on the proof of certain great theorems on this question. In 1912 he proved the hairy ball theorem for the two-dimensional sphere, as well as the fact that every continuous map from the two-dimensional ball to itself has a fixed point.[26] . These two results in themselves were not really new. As Hadamard observed, Poincaré had shown a theorem equivalent to the hairy ball theorem[27] . The revolutionary aspect of Brouwer's approach was his systematic use of recently developed tools such as homotopy, the underlying concept of the Poincaré group. In the following year, Hadamard generalised the theorem under discussion to an arbitrary finite dimension, but he employed different methods. H. Freudenthal comments on the respective roles as follows: "Compared to Brouwer's revolutionary methods, those of Hadamard were very traditional, but Hadamard's participation in the birth of Brouwer's ideas resembles that of a midwife more than that of a mere spectator."[28] .

Brouwer's approach yielded its fruits, and in 1912 he also found a proof that was valid for any finite dimension.[29] , as well as other key theorems such as the invariance of dimension[30] . In the context of this work, Brouwer also generalized the Jordan curve theorem to arbitrary dimension and established the properties connected with the degree of a continuous mapping.[31] . This branch of mathematics, originally envisioned by Poincaré and developed by Brouwer, changed its name. In the 1930s, analysis situs became → algebraic topology.[32] .

Brouwer's celebrity is not exclusively due to his topological work. He was also the originator and zealous defender of a way of formalising mathematics that is known as intuitionism, which at the time made a stand against set theory.[33] . While Brouwer preferred constructive proofs, ironically, the original proofs of his great topological theorems were not constructive[34] , and it took until 1967 for constructive proofs to be found.[35]

Reception

John Nash used the theorem in game theory to prove the existence of a winning strategy.

The theorem proved its worth in more than one way. During the 20th century numerous fixed point theorems were developed, and even a branch of mathematics called fixed point theory.[36] Brouwer's theorem is probably the most important.[37] It is also among the foundational theorems on the topology of topological manifolds and is often used to prove other important results such as the Jordan curve theorem.[38]

Besides the fixed point theorems for more or less contracting functions, there are many that have emerged directly or indirectly from the result under discussion. A continuous map from a closed ball of Euclidean space to its boundary cannot be the identity on the boundary. Similarly, the Borsuk–Ulam theorem says that a continuous map from the n-dimensional sphere to $\mathbf{R}^n$ has a pair of antipodal points that are mapped to the same point. In the finite-dimensional case, the Lefschetz fixed point theorem provided from 1926 a method for counting fixed points. In 1930, Brouwer's fixed point theorem was generalized to Banach spaces.[39] . This generalization is known as Schauder's fixed point theorem, a result generalized further by S. Kakutani to multivalued functions.[40] One also meets the theorem and its variants outside topology. It can be used to prove the Hartman-Grobman theorem, which describes the qualitative behaviour of certain differential equations near certain equilibria. Similarly, Brouwer's theorem is used for the proof of the théorème de la variété centrale. The theorem can also be found in existence proofs for the solutions of certain partial differential equations.[41]

Other areas are also touched. In game theory, John Nash used the theorem to prove that in the game of Hex there is a winning strategy for white.[42] . In economy, P. Bich explains that certain generalizations of the theorem show that its use is helpful for certain classical problems in game theory and generally for equilibria (Hotelling's law), financial equilibria and incomplete markets.[43]

Proof outlines

A full proof of the theorem would be too long to reproduce here, but the following outlines a proof omitting the most difficult part. The proof uses the observation that the boundary of D^n is S^{n-1}, the $(n-1)$-sphere.

The argument proceeds by contradiction, supposing that a continuous function $f : D^n \to D^n$ has *no* fixed point, and then attempting to derive an inconsistency, which proves that the function must in fact have a fixed point. For each x in D^n, there is only one straight line that passes through $f(x)$ and x, because it must be the case that $f(x)$ and x are distinct by hypothesis (recall that f having no fixed points means that $f(x) \neq x$). Following this line from $f(x)$ through x leads to a point on S^{n-1}, denoted by $F(x)$. This defines a continuous function $F : D^n \to S^{n-1}$, which is a special type of continuous function known as a retraction: every point of the codomain (in this case S^{n-1}) is a fixed point of the function.

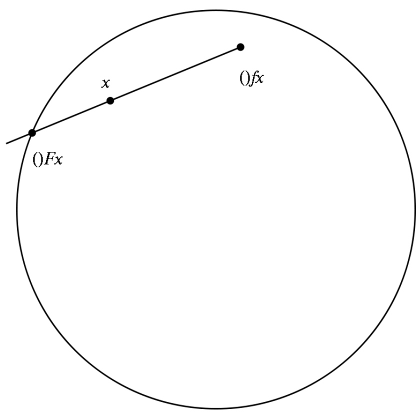


Illustration of the retraction F

Intuitively it seems unlikely that there could be a retraction of D^n onto S^{n-1}, and in the case $n = 1$ it is obviously impossible because S^0 (i.e., the endpoints of the closed interval D^1) is not even connected. The case $n = 2$ is less obvious, but can be proven by using basic arguments involving the → fundamental groups of the respective spaces: the retraction would induce an injective group homomorphism from the fundamental group of S^1 to that of D^2, but the first group is isomorphic to **Z** while the latter group is trivial, so this is impossible. The case $n = 2$ can also be proven by contradiction based on a theorem about non-vanishing vector fields.

For $n > 2$, however, proving the impossibility of the retraction is more difficult. One way is to make use of homology groups: it can be shown that the homology $H_{n-1}(D^n)$ is trivial, while $H_{n-1}(S^{n-1})$ is infinite cyclic. This shows that the retraction is impossible, because again the retraction would induce an injective group homomorphism from the latter to the former group.

There is also a more elementary combinatorial proof, whose main step consists in establishing Sperner's lemma in n dimensions.

There is also a quick proof, by Morris Hirsch, based on the impossibility of a differentiable retraction. The indirect proof starts by noting that the map f can be approximated by a smooth map retaining the property of not fixing a point; this can be done by using the Weierstrass approximation theorem, for example. One then defines a retraction as above which must now be differentiable. Such a retraction must have a non-singular value, by Sard's theorem, which is also non-singular for the restriction to the boundary (which is just the identity). Thus the inverse image would be a 1-manifold with boundary. The boundary would have to contain at least two end points, both of which would have to lie on the boundary of the original ball—which is impossible in a retraction.

A quite different proof given by David Gale is based on the game of Hex. The basic theorem about Hex is that no game can end in a draw. This is equivalent to the Brouwer fixed point theorem for dimension 2. By considering n-dimensional versions of Hex, one can prove in general that Brouwer's theorem is equivalent to the determinacy theorem for Hex. [44]

Generalizations

The Brouwer fixed point theorem forms the starting point of a number of more general fixed point theorems.

The straightforward generalization to infinite dimensions, i.e. using the unit ball of an arbitrary Hilbert space instead of Euclidean space, is not true. The main problem here is that the unit balls of infinite-dimensional Hilbert spaces are not compact. For example, in the Hilbert space ℓ^2 of square-summable real (or complex) sequences, consider the map $f : \ell^2 \to \ell^2$ which sends a sequence (x_n) from the closed unit ball of ℓ^2 to the sequence (y_n) defined by

$$y_0 = \sqrt{1 - \|x\|_2^2} \qquad \text{and} \qquad y_n = x_{n-1} \quad \text{for} \quad n \geq 1.$$

It is not difficult to check that this map is continuous, has its image in the unit sphere of ℓ^2, but does not have a fixed point.

The generalizations of the Brouwer fixed point theorem to infinite dimensional spaces therefore all include a compactness assumption of some sort, and in addition also often an assumption of convexity. See fixed point theorems in infinite-dimensional spaces for a discussion of these theorems.

The Kakutani fixed point theorem generalizes the Brouwer fixed point theorem in a different direction: it stays in $\mathbf{R}^n$, but considers upper semi-continuous correspondences (functions that assign to each point of the set a subset of the set). It also requires compactness and convexity of the set.

By using forcing to collapse cardinals, the Brouwer fixed point theorem can be generalized to arbitrary cardinality: if L is a compact, connected order topology, then any continuous function from L^n to itself has a fixed point. Note that if we require L to be separable, this is precisely the Brouwer fixed point theorem.

The Lefschetz fixed-point theorem applies to (almost) arbitrary compact topological spaces, and gives a condition in terms of singular homology that guarantees the existence of fixed points; this condition is trivially satisfied for any

map in the case of D^n.

See also

- Tucker's lemma
- Topological combinatorics

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- S. Karamadian (ed.), *Fixed points. Algorithms and applications*, Academic Press, 1977
- V.I. Istrăţescu, *Fixed point theory*, Reidel, 1981

External links

- Brouwer's Fixed Point Theorem for Triangles [47] at cut-the-knot
- Brouwer theorem [48], from PlanetMath with attached proof.
- Reconstructing Brouwer [49] at MathPages

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[1] E.g. F & V Bayart *Théorèmes du point fixe* (http://www.bibmath.net/dico/index.php3?action=affiche&quoi=./p/pointfixe.html) on Bibm@th.net

[2] See page 15 of: D. Leborgne *Calcul différentiel et géométrie* Puf (1982) ISBN 2130374956

[3] More exactly, according to Encyclopédie Universalis: *Il en a démontré l'un des plus beaux théorèmes, le théorème du point fixe, dont les applications et généralisations, de la théorie des jeux aux équations différentielles, se sont révélées fondamentales.* Luizen Brouwer (http://www.universalis.fr/encyclopedie/T705705/BROUWER_L.htm) by G. Sabbagh

[4] D. Violette *Applications du lemme de Sperner pour les triangles* (http://newton.mat.ulaval.ca/amq/bulletins/dec06/sperner.pdf) Bulletin AMQ, V. XLVI N° 4, (2006) p 17.

[5] Page 15 of: D. Leborgne *Calcul différentiel et géométrie* Puf (1982) ISBN 2130374956.

[6] This version follows directly from the previous one because every convex compact subset of a Euclidean space is homeomorphic to a closed ball. For details see convex set.

[7] V. & F. Bayart *Point fixe, et théorèmes du point fixe* (http://www.bibmath.net/dico/index.php3?action=affiche&quoi=./p/pointfixe.html) on Bibmath.net.

[8] C. Minazzo K. Rider *Théorèmes du Point Fixe et Applications aux Equations Différentielles* (http://math1.unice.fr/~eaubry/Enseignement/M1/memoire.pdf) Université de Nice-Sophia Antipolis.

[9] The interest of this anecdote rests in its intuitive and didactic character, but its accuracy is dubious. As the history section shows, the origin of the theorem is not Brouwer's work. More than 20 years earlier Henri Poincaré had proved an equivalent results, and 5 years before Brouwer P. Bohl had proved the three-dimensional case.

[10] Cette citation provient d'une émission de télévision : *Archimède* (http://archives.arte.tv/hebdo/archimed/19990921/ftext/sujet5.html), Arte, 21 septembre 1999

[11] See F. Brechenmacher *L'identité algébrique d'une pratique portée par la discussion sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes* (http://arxiv.org/ftp/arxiv/papers/0704/0704.2931.pdf) CNRS Fédération de Recherche Mathématique du Nord-Pas-de-Calais

[12] Henri Poincaré won the King of Sweden's mathematical competition in 1889 for his work on the related three-body problem: J. Tits *Célébrations nationales 2004* (http://www.culture.gouv.fr/culture/actualites/celebrations2004/poincare.htm) Site du Ministère Culture et Communication

[13] H. Poincaré *Les méthodes nouvelles de la mécanique céleste* T Gauthier-Villars, Vol 3 p 389 (1892) new edition Paris: Blanchard, 1987.

[14] Quotation from H. Poincaré taken from: P. A. Miquel *La catégorie de désordre* (http://www.arches.ro/revue/no03/no3art03.htm), on the website of l'Association roumaine des chercheurs francophones en sciences humaines

[15] This question was studied in: H. Poincaré *Sur les courbes définies par les équations différentielles* J. de Math. V 2 (1886)

[16] This follows from the Poincaré–Bendixson theorem.

[17] Multiplication by $\frac{1}{2}$ on $]0, 1[^2$ has no fixed point.

[18] "concerne les propriétés invariantes d'une figure lorsqu'on la déforme de manière continue quelconque, sans déchirure (par exemple, dans le cas de la déformation de la sphère, les propriétés corrélatives des objets tracés sur sa surface". From C. Houzel M. Paty *Poincaré, Henri (1854-1912)* (http://www.scientiaestudia.org.br/associac/paty/pdf/Paty,M_1997g-PoincareEU.pdf) Encyclopædia Universalis Albin Michel, Paris, 1999, p. 696-706

[19] Poincaré's theorem is stated in: V. I. Istratescu *Fixed Point Theory an Introduction* Kluwer Academic Publishers (réédition de 2001) p 113 ISBN 1402003013

[20] M.I. Voitsekhovskii *Brouwer theorem* (http://eom.springer.de/b/b017670.htm) Encyclopaedia of Mathematics ISBN 1402006098

[21] J. Dieudonné, *A History of Algebraic and differential Topology, 1900-1960*, pages 17-24

[22] See for example: E Picard *Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires* (http://portail.mathdoc.fr/JMPA/PDF/JMPA_1893_4_9_A4_0.pdf) Journal de Mathématiques p 217 (1893)

[23] J. J. O'Connor E. F. Robertson *Piers Bohl* (http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Bohl.html)

[24] A. D. Myskis I. M. Rabinovic *The first proof of a fixed-point theorem for a continuous mapping of a sphere into itself, given by the Latvian mathematician P G Bohl (Russian)* Uspekhi matematicheskikh nauk (NS) Vol 10 (N° 3) (65) (1955) pp 188-192

[25] J. J. O'Connor E. F. Robertson *Luitzen Egbertus Jan Brouwer* (http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Brouwer.html)

[26] H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 495 (1975)

[27] Freudenthal explains: "... cette dernière propriété, bien que sous des hypothèses plus grossières, ait été démontré par H. Poincaré" H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 495 (1975)

[28] H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 501 (1975)

[29] L. Brouwer *Über Abbildungen von Mannigfaltigkeiten* Mathematische Annalen 38(71) pp 97 115 (1912)

[30] If an open subset of a manifold is homeomorphic to an open subset of a Euclidean space of dimension n, and if p is a positive integer other than n, then the open set is never homeomorphic to an open subset of a Euclidean space of dimension p.

[31] J. J. O'Connor E. F. Robertson *Luitzen Egbertus Jan Brouwer* (http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Brouwer.html).

[32] The term *algebraic topology* first appeared 1931 under the pen of David van Dantzig: J. Miller *Topological algebra* (http://jeff560.tripod.com/t.html) on the site Earliest Known Uses of Some of the Words of Mathematics (2007)

[33] Later it would be shown that the formalism that was combatted by Brouwer can also serve to formalise intuitionism. For further details see intuitionistic logic.

[34] For a long explanation, see: J.P. Dubucs *L.J.E. Brouwer : Topologie et constructivisme* (http://www.persee.fr/web/revues/home/prescript/article/rhs_0151-4105_1988_num_41_2_4094) Revue d'histoire des sciences V. 41 N°41-2 pp 133-155 (1988)

[35] H. Scarf found the first algorithmic proof: M.I. Voitsekhovskii *Brouwer theorem* (http://eom.springer.de/b/b017670.htm) Encyclopaedia of Mathematics ISBN 1402006098.

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Cellular approximation theorem

In → algebraic topology, in the **cellular approximation theorem**, a map between CW-complexes can always be taken to be of a specific type. Concretely, if X and Y are CW-complexes, and $f : X \to Y$ is a continuous map, then f is said to be *cellular*, if f takes the n-skeleton of X to the n-skeleton of Y for all n, i.e. if $f(X^n) \subseteq Y^n$ for all n. The content of the cellular approximation theorem is then that any continuous map $f : X \to Y$ between CW-complexes X and Y is homotopic to a cellular map, and if f is already cellular on a subcomplex A of X, then we can furthermore choose the homotopy to be stationary on A. From an algebraic topological viewpoint, any map between CW-complexes can thus be taken to be cellular.

Idea of proof

The proof can be given by induction after n, with the statement that f is cellular on the skeleton X^n. For the base case $n = 0$, notice that every path-component of Y must contain a 0-cell. The image under f of a 0-cell of X can thus be connected to a 0-cell of Y by a path, but this gives a homotopy from f to a map, which is cellular on the 0-skeleton of X.

Assume inductively that f is cellular on the $(n - 1)$-skeleton of X, and let e^n be an n-cell of X. The closure of e^n is compact in X, being the image of the characteristic map of the cell, and hence the image of the closure of e^n under f is also compact in Y. Then it is a general result of CW-complexes that any compact subspace of a CW-complex meets (that is, intersects non-trivially) only finitely many cells of the complex. Thus $f(e^n)$ meets at most finitely many cells of Y, so we can take $e^k \subseteq Y$ to be a cell of highest dimension meeting $f(e^n)$. If $k \leq n$, the map f is already cellular on e^n, since in this case only cells of the n-skeleton of Y meets $f(e^n)$, so we may assume that $k > n$. It is then a technical, non-trivial result (see Hatcher) that the restriction of f to $X^{n-1} \cup e^n$ can be homotoped relative to X^{n-1} to a map missing a point $p \in e^k$. Since $Y^k - \{p\}$ deformation retracts onto the subspace $Y^k - e^k$, we can further homotope the restriction of f to $X^{n-1} \cup e^n$ to a map, say, g, with the property that $g(e^n)$ misses the cell e^k of Y, still relative to X^{n-1}. Since $f(e^n)$ met only finitely many cells of Y to begin with, we can repeat this process finitely many times to make $f(e^n)$ miss all cells of Y of dimension larger than n.

We repeat this process for every n-cell of X, fixing cells of the subcomplex A on which f is already cellular, and we thus obtain a homotopy (relative to the $(n - 1)$-skeleton of X and the n-cells of A) of the restriction of f to X^n to a map cellular on all cells of X of dimension at most n. Using then the homotopy extension property to extend this to a homotopy on all of X, and patching these homotopies together, will finish the proof. For details, consult Hatcher.

Applications

Some homotopy groups

The cellular approximation theorem can be used to immediately calculate some homotopy groups. In particular, if $n < k$, then $\pi_n(S^k) = 0$: Give S^n and S^k their canonical CW-structure, with one 0-cell each, and with one n-cell for S^n and one k-cell for S^k. Any base-point preserving map f: $S^n \to S^k$ is by the cellular approximation theorem homotopic to a constant map, whence $\pi_n(S^k) = 0$.

Cellular approximation for pairs

Let $f{:}(X,A)\to(Y,B)$ be a map of CW-pairs, that is, f is a map from X to Y, and the image of $A \subseteq X$ under f sits inside B. Then f is homotopic to a cellular map $(X,A)\to(Y,B)$. To see this, restrict f to A and use cellular approximation to obtain a homotopy of f to a cellular map on A. Use the homotopy extension property to extend this homotopy to all of X, and apply cellular approximation again to obtain a map cellular on X, but without violating the cellular property on A.

As a consequence, we have that a CW-pair *(X,A)* is n-connected, if all cells of $X - A$ have dimension strictly greater than *n*:
If $i \leq n$, then any map $(D^i, \partial D^i)$→*(X,A)* is homotopic to a cellular map of pairs, and since the *n*-skeleton of *X* sits inside *A*, such map is homotopic to a map whose image is in *A*, and hence it is 0 in the relative homotopy group $\pi_i(X, A)$.
We have in particular that $(X, X^n)$is *n*-connected, so it follows from the long exact sequence of homotopy groups for the pair $(X,$ that we have isomorphisms $\pi_i(X^n)$→ $\pi_i(X)$ for all $i < n$ and a surjection $\pi_n(X^n)$→ $\pi_n(X)$.

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Eilenberg–Zilber theorem

In mathematics, specifically in → algebraic topology, the **Eilenberg–Zilber theorem** is an important result in establishing the link between the homology groups of a product space $X \times Y$and those of the spaces X and Y. The theorem first appeared in a 1953 paper in the American Journal of Mathematics.

Statement of the theorem

The theorem can be formulated as follows. Suppose X and $Y$are topological spaces, Then we have the three chain complexes $C_*(X)$, $C_*(Y)$, and $C_*(X \times Y)$. (The argument applies equally to the simplicial or singular chain complexes.) We also have the *tensor product complex* $C_*(X) \otimes C_*(Y)$, whose differential is, by definition,

$$\delta(\sigma \otimes \tau) = \delta_X \sigma \otimes \tau + (-1)^p \sigma \otimes \delta_Y \tau$$

for $\sigma \in C_p(X)$and δ_X, $\delta_Y$the differentials on $C_*(X)$, $C_*(Y)$.
Then the theorem says that we have a chain maps

$$F : C_*(X \times Y) \to C_*(X) \otimes C_*(Y), \quad G : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

such that FG is the identity and GF is chain-homotopic to the identity. Moreover, the maps are natural in X and Y. Consequently the two complexes must have the same homology:

$$H_*(C_*(X \times Y)) \cong H_*(C_*(X) \otimes C_*(Y)).$$

An important generalisation to the nonabelian case using crossed complexes is given in the paper by Tonks below. This give full details of a result on the (simplicial) classifying space of a crossed complex stated but not proved in the paper by Brown and Higgins on classifying spaces.

Consequences

The Eilenberg–Zilber theorem is a key ingredient in establishing the Künneth theorem, which expresses the homology groups $H_*(X \times Y)$in terms of $H_*(X)$and $H_*(Y)$. In light of the Eilenberg–Zilber theorem, the content of the Künneth theorem consists in analysing how the homology of the tensor product complex relates to the homologies of the factors; the answer is somewhat subtle.

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Hurewicz theorem

In mathematics, the **Hurewicz theorem** is a basic result of → algebraic topology, connecting → homotopy theory with → homology theory via a map known as the **Hurewicz homomorphism**. The theorem is named after → Witold Hurewicz, and generalizes earlier results from Henri Poincaré.

Statement of the theorems

The Hurewicz theorems are a key link between homotopy groups and homology groups.

Absolute version

For any space *X* and positive integer *k* there exists a group homomorphism

$$h_*\colon \pi_k(X) \to H_k(X)$$

called the Hurewicz homomorphism from the *k*-th homotopy group to the *k*-th homology group (with integer coefficients), which for $k = 1$ is equivalent to the canonical abelianization map

$$h_*\colon \pi_1(X) \to \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

The Hurewicz theorem states that if *X* is $(n-1)$-connected, the Hurewicz map is an isomorphism for all $k \le n$. In particular, this theorem says that the abelianization of the first homotopy group (the → fundamental group) is isomorphic to the first homology group:

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

The first homology group therefore vanishes if *X* is path-connected and $\pi_1(X)$ is a perfect group.

In addition, the Hurewicz homomorphism is an epimorphism from $\pi_{n+1}(X) \to H_{n+1}(X)$ whenever X is (n-1) connected, for $n \geq 2$.

The group homomorphism is given in the following way. Choose canonical generators $u_n \in H_n(S^n)$. Then a homotopy class of maps $f \in \pi_n(X)$ is taken to $f_*(u_n) \in H_n(X)$.

Relative version

For any pair of spaces (X,A) and integer $k > 1$ there exists a homomorphism

$$h_*\colon \pi_k(X,A) \to H_k(X,A)$$

from relative homotopy groups to relative homology groups. The Relative Hurewicz Theorem states that if each of *X*, *A* are connected and the pair (X,A) is $(n-1)$-connected then $H_k(X,A) = 0$ for $k < n$ and $H_n(X,A)$ is obtained from $\pi_n(X,A)$ by factoring out the action of $\pi_1(A)$. This is proved in, for example, Whitehead (1978) by induction, proving in turn the absolute version and the Homotopy Addition Lemma.

This relative Hurewicz theorem is reformulated by Brown & Higgins (1981) as a statement about the morphism

$$\pi_n(X, A) \to \pi_n(X \cup CA).$$

This statement is a special case of a homotopical excision theorem, involving induced modules for n>2 (crossed modules if n=2), which itself is deduced from a higher homotopy van Kampen theorem for relative homotopy groups, whose proof requires development of techniques of a cubical higher homotopy groupoid of a filtered space.

Triadic version

For any triad of spaces $(X;A,B)$ (i.e. space X and subspaces A,B) and integer $k > 2$ there exists a homomorphism

$$h_*: \pi_k(X; A, B) \to H_k(X; A, B)$$

from triad homotopy groups to triad homology groups. Note that $H_k(X;A,B) \cong H_k(X \cup (C(A \cup B)))$. The Triadic Hurewicz Theorem states that if X, A, B, and $C = A \cap B$ are connected, the pairs (A,C), (B,C) are respectively $(p-1)$-, $(q-1)$-connected, and the triad $(X;A,B)$ is $p+q-2$ connected, then $H_k(X;A,B) = 0$ for $k < p+q-2$ and $H_{p+q-1}(X;A)$ is obtained from $\pi_{p+q-1}(X;A,B)$ by factoring out the action of $\pi_1(A \cap B)$ and the generalised Whitehead products. The proof of this theorem uses a higher homotopy van Kampen type theorem for triadic homotopy groups, which requires a notion of the fundamental cat^n-group of an n-cube of spaces.

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Kunneth theorem

In mathematics, especially in → homological algebra and → algebraic topology, a **Künneth theorem** is a statement relating the homology of two objects to the homology of their product. The classical statement of the Künneth theorem relates the singular homology of two topological spaces X and Y and their product space $X \times Y$. In the simplest possible case the relationship is that of a tensor product, but for applications it is very often necessary to apply certain tools of homological algebra to express the answer.

A Künneth theorem or Künneth formula is true in many different homology and cohomology theories, and the name has become generic. These many results are named for the German mathematician → Otto Hermann Künneth (1892–1975).

Singular homology with coefficients in a field

Let X and Y be two topological spaces, and let F be a field. In this situation, the Künneth theorem for singular homology states that for any integer k,

$$\bigoplus_{i+j=k} H_i(X,F) \otimes H_j(Y,F) \cong H_k(X \times Y, F).$$

Furthermore, the isomorphism is natural isomorphism. The map from the sum to the homology group of the product is called the *cross product*. More precisely, there is a cross product operation showing how an i-cycle on X and a j-cycle on Y can be combined to create an $(i + j)$-cycle on $X \times Y$; so that there is an explicit linear mapping defined from the direct sum to $H_k(X \times Y)$.

A consequence of this result is that the Betti numbers, the dimensions of the homology with **Q** coefficients, of $X \times Y$ can be determined from those of X and Y. If $p_Z(t)$ is the generating function of the sequence of Betti numbers $b_k(Z)$ of a space Z, then

$$p_{X\times Y}(t) = p_X(t)p_Y(t).$$

Here when there are finitely many Betti numbers of X and Y, each of which is a natural number rather than ∞, this reads as an identity on Poincaré polynomials. In the general case these are formal power series with possibly infinite coefficients, and have to be interpreted accordingly. Furthermore, the above statement holds not only for the Betti numbers but also for the generating functions of the dimensions of the homology over any field. (If the integer homology is not torsion-free then these numbers may be different.)

Singular homology with coefficients in a PID

The above formula is simple because vector spaces over a field have very restricted behavior. As the coefficient ring becomes more general, the relationship becomes more complicated. The next simplest case is the case when the coefficient ring is a principal ideal domain. This case is particularly important because the integers are a PID.

In this case the equation above is no longer true. Instead a correction factor appears to account for the possibility of torsion phenomena. For example, if $H_1(X,\mathbf{Z}) = \mathbf{Z}/(2)$ and $H_1(Y,\mathbf{Z}) = \mathbf{Z}/(3)$, then the tensor product of these homology groups will be zero. But the second homology of $X \times Y$ will *always* contain a correction factor to account for the vanishing of this product. This correction factor is expressed in terms of the Tor functor, the first derived functor of the tensor product.

When X and Y are CW complexes and R is a PID, then the correct statement of the Künneth theorem is that there are natural short exact sequences

$$0 \to \bigoplus_{i+j=k} H_i(X,R) \otimes_R H_j(Y,R) \to H_k(X \times Y, R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(X,R), H_j(Y,R)) \to 0.$$

Furthermore these sequences split, but not canonically.

The Künneth spectral sequence

For general *R*, the homology of *X* and *Y* is related to the homology of their product by a spectral sequence. In the cases described above, this spectral sequence collapses to give an isomorphism or a short exact sequence. The Künneth spectral sequence is

$$E_{pq}^2 = \bigoplus_{q_1+q_2=q} \operatorname{Tor}_p^R(H_{q_1}(X,R), H_{q_2}(Y,R)) \Rightarrow H_{p+q}(X \times Y, R).$$

The Künneth formula in the derived category

A much cleaner statement of the Künneth formula becomes possible in the derived category. In this case, the formula becomes a natural isomorphism between quasi-isomorphism classes of singular chain complexes:

$$C_*(X) \otimes^{\mathbf{L}} C_*(Y) \cong C_*(X \times Y).$$

Here $\otimes^{\mathbf{L}}$denotes the derived tensor product.

Künneth theorems in other homology and cohomology theories

The above statements are also true for singular cohomology and sheaf cohomology. For sheaf cohomology on an algebraic variety, Grothendieck found six spectral sequences relating the possible hyperhomology groups of two chain complexes of sheaves and the hyperhomology groups of their tensor product. (See EGA III_2, Théorème 6.7.3) Künneth theorems are also true for → K-theory, cobordism, and l-adic cohomology.

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Poincaré duality theorem

In mathematics, the **Poincaré duality** theorem, named after Henri Poincaré, is a basic result on the structure of the homology and cohomology groups of manifolds. It states that if M is an n-dimensional oriented closed manifold (compact and without boundary), then the kth cohomology group of M is isomorphic to the $(n - k)$th homology group of M, for all integers k: $H^k(M) \cong H_{n-k}(M)$.

Poincaré duality holds for any coefficient ring, so long as one has taken an orientation with respect to that coefficient ring; in particular, since every manifold has a unique orientation mod 2, Poincaré duality holds mod 2 without any assumption of orientation.

History

A form of Poincaré duality was first stated, without proof, by Henri Poincaré in 1893. It was stated in terms of Betti numbers: The kth and $(n - k)$ th Betti numbers of a closed (i.e. compact and without boundary) orientable n-manifold are equal. The *cohomology* concept was at that time about 40 years from being clarified. In his 1895 paper *Analysis Situs*, Poincaré tried to prove the theorem using topological intersection theory, which he had invented. Criticism of his work by Poul Heegaard led him to realize that his proof was seriously flawed. In the first two complements to *Analysis Situs*, Poincaré gave a new proof in terms of dual triangulations.

Poincaré duality did not take on its modern form until the advent of cohomology in the 1930s, when Eduard Čech and → Hassler Whitney invented the cup and cap products and formulated Poincaré duality in these new terms.

Modern formulation

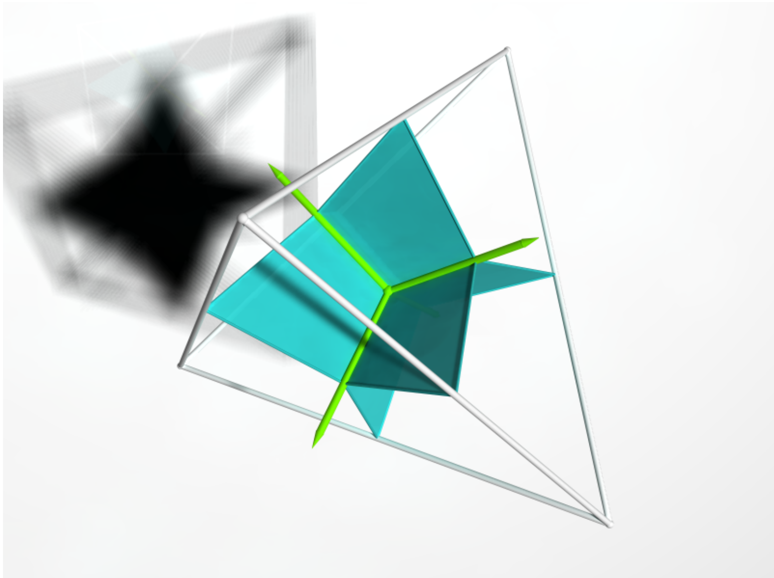
The modern statement of the Poincaré duality theorem is in terms of homology and cohomology: if M is a closed oriented n-manifold, and k is an integer, then there is a canonically defined isomorphism from the k-th homology group $H_k(M)$ to the $(n - k)$th cohomology group $H^{n-k}(M)$. (Here, homology and cohomology is taken with coefficients in the ring of integers, but the isomorphism holds for any coefficient ring.) Specifically, one maps an element of $H^k(M)$ to its cap product with a fundamental class of M, which will exist for oriented M.

For non-compact oriented manifolds, one has to replace cohomology by cohomology with compact support.

Homology and cohomology groups are defined to be zero for negative degrees, so Poincaré duality in particular implies that the homology and cohomology groups of orientable closed n-manifolds are zero for degrees bigger than n.

Dual cell structures

Given a triangulated manifold, there is a corresponding dual polyhedral decomposition. The dual polyhedral decomposition is a cell decomposition of the manifold such that the *k*-cells of the dual polyhedral decomposition are in bijective correspondence with the *n*-*k*-cells of the triangulation, generalising the notion of dual polyhedra.



$\cup_{S \in T} \Delta \cap DS$ -- a picture of the parts of the dual-cells in a top-dimensional simplex.

Precisely, let *T* be a triangulation of an *n*-manifold *M*. Let *S* be a simplex of *T*. We denote the dual cell (to be defined precisely) corresponding to *S* by *DS*. Let Δ be a top-dimensional simplex of *T* containing *S*. So we can think of *S* as a subset of the vertices of Δ. Then $\Delta \cap DS$ is defined to be the convex hull (in Δ) of the barycentres of all subsets of the vertices of Δ that contain S. One can check that if *S* is *i*-dimensional, then *DS* is an *n*-*i*-dimensional cell. Moreover, the dual cells to *T* form a CW-decomposition of *M*, and the only *n*-*i*-dimensional dual cell that intersects an *i*-cell *S* is *DS*. Thus the pairing $C_i M \otimes C_{n-i} M \to \mathbb{Z}$ given by taking intersections induces an isomorphism $C_i M \to C^{n-1} M$, where here C_i is the cellular homology of the triangulation *T*, and $C_{n-i} M$ and $C^{n-i} M$ are the cellular homologies and cohomologies of the dual polyhedral/CW decomposition the manifold respectively. The fact that this is an isomorphism of chain complexes is a proof of Poincare Duality. Roughly speaking, this amounts to the fact that the boundary relation for the triangulation *T* is the incidence relation for the dual polyhedral decomposition under the correspondence $S \longmapsto DS$.

Naturality

Note that H^k is a contravariant functor while H_{n-k} is covariant. The family of isomorphisms

$$D_M : H^k(M) \to H_{n-k}(M)$$

is natural in the following sense: if

$$f : M \to N$$

is a continuous map between two oriented *n*-manifolds which is compatible with orientation, i.e. which maps the fundamental class of *M* to the fundamental class of *N*, then

$$D_N = f_* D_M f^*,$$

where f_* and f^* are the maps induced by *f* in homology and cohomology, respectively.

Bilinear pairings formulation

Assuming *M* is compact boundaryless and orientable, let $\tau H_i M$ denote the torsion subgroup of $H_i M$ and let $f H_i M = H_i M / \tau H_i M$ be the free part – all homology groups taken with integer coefficients in this section. Then there are bilinear maps which are duality pairings

$$fH_iM \otimes fH_{n-i}M \to \mathbb{Z}$$

and

$$\tau H_i M \otimes \tau H_{n-i-1} M \to \mathbb{Q}/\mathbb{Z}.$$

(Here $\mathbb{Q}/\mathbb{Z}$ is the quotient of the rationals by the integers, taken as an additive group.)

(Notice that in the torsion linking form, there is a −1 in the dimension, so the paired dimensions add up to $n-1$, rather than to n.)

The first form is typically called the *intersection product* and the 2nd the *torsion linking form*. Assuming the manifold *M* is smooth, the intersection product is computed by perturbing the homology classes to be transverse and computing their oriented intersection number. For the torsion linking form, one computes the pairing of *x* and *y* by realizing *nx* as the boundary of some class *z*. The form is the fraction with numerator the transverse intersection number of *z* with *y* and denominator *n*.

The statement that the pairings are duality pairings means that the adjoint maps

$$fH_iM \to \mathrm{Hom}_{\mathbb{Z}}(fH_{n-i}M, \mathbb{Z})$$

and

$$\tau H_iM \to \mathrm{Hom}_{\mathbb{Z}}(\tau H_{n-i-1}M, \mathbb{Q}/\mathbb{Z})$$

are isomorphisms of groups.

This result is an application of Poincare Duality $H_iM \simeq H^{n-i}M$ together with the → Universal coefficient theorem which gives an identification $fH^{n-i}M \equiv \mathrm{Hom}(H_{n-i}M;\mathbb{Z})$ and $\tau H^{n-i}M \equiv \mathrm{Ext}(H_{n-i-1}M;\mathbb{Z}) \equiv \mathrm{Hom}(\tau H_{n-i-1}M;\mathbb{Q}/\mathbb{Z})$. Thus, Poincaré duality says that fH_iM and $fH_{n-i}M$ are isomorphic, although there is no natural map giving the isomorphism, and similarly τH_iM and $\tau H_{n-i-1}M$ are also isomorphic, though not naturally.

This approach to Poincaré duality was used by Przytycki and Yasuhara to give an elementary homotopy and diffeomorphism classification of 3-dimensional lens spaces.[1]

Generalizations and related results

The Poincaré-Lefschetz duality theorem is a generalisation for manifolds with boundary. In the non-orientable case, taking into account the sheaf of local orientations, one can give a statement that is independent of orientability.

Blanchfield duality is a version of Poincaré duality which provides an isomorphism between the homology of an abelian covering space of a manifold and the corresponding cohomology with compact supports. It is used to get basic structural results about the Alexander module and can be used to define the signatures of a knot.

With the development of → homology theory to include → K-theory and other *extraordinary* theories from about 1955, it was realised that the homology H_* could be replaced by other theories, once the products on manifolds were constructed; and there are now textbook treatments in generality.

Verdier duality is the appropriate generalization to (possibly singular) geometric objects, such as analytic spaces or schemes, while intersection homology was developed R. MacPherson and M. Goresky for stratified spaces, such as real or complex algebraic varieties, precisely so as to generalise Poincaré duality to such stratified spaces.

There are many other forms of geometric duality in → algebraic topology, including Lefschetz duality, Alexander duality, Hodge duality, and S-duality (homotopy theory).

See also

- Bruhat decomposition
- Fundamental class
- Weyl group

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Universal coefficient theorem

In mathematics, the **universal coefficient theorem** in → algebraic topology establishes the relationship in → homology theory between the *integral homology* of a topological space X, and its *homology with coefficients* in any abelian group A. It states that the integral homology groups

$$H_i(X, \mathbb{Z})$$

completely determine the groups

$$H_i(X, A).$$

Here H_i might be the simplicial homology or more general singular homology theory: the result itself is a pure piece of → homological algebra about chain complexes of free abelian groups. The form of the result is that other coefficients A may be used, at the cost of using a Tor functor.

For example it is common to take A to be $\mathbb{Z}/2\mathbb{Z}$, so that coefficients are modulo 2. This becomes straightforward in the absence of 2-torsion in the homology. Quite generally, the result indicates the relationship that holds between the Betti numbers b_i of X and the Betti numbers $b_{i,F}$ with coefficients in a field F. These can differ, but only when the characteristic of F is a prime number p for which there is some p-torsion in the homology.

Statement

Consider the tensor product $H_i(X, \mathbb{Z}) \otimes A$. The theorem states that there is an injective group homomorphism ι from this group to $H_i(X, A)$, which has cokernel $Tor(H_{i-1}(X, \mathbb{Z}), A)$.

In other words, there is a natural short exact sequence

$$0 \to H_i(X, \mathbb{Z}) \otimes A \to H_i(X, A) \to \mathrm{Tor}(H_{i-1}(X, \mathbb{Z}), A) \to 0.$$

Furthermore, this is a split sequence (but the splitting is *not* natural).

The Tor group on the right can be thought of as the obstruction to ι being an isomorphism.

Universal coefficient theorem for cohomology

There is also a **universal coefficient theorem for cohomology** involving the Ext functor, stating that there is a natural short exact sequence

$$0 \to \mathrm{Ext}(H_{i-1}(X,\mathbb{Z}),A) \to H^i(X,A) \to \mathrm{Hom}(H_i(X,\mathbb{Z}),A) \to 0.$$

As in the homological case, the sequence splits, though not naturally.

References

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Van Kampen's theorem

In mathematics, the **Seifert–van Kampen theorem** of → algebraic topology, sometimes just called **van Kampen's theorem**, expresses the structure of the → fundamental group of a topological space X, in terms of the fundamental groups of two open, path-connected subspaces U and V that cover X. It can therefore be used for computations of the fundamental group of spaces that are constructed out of simpler ones.

The underlying idea is that paths in X can be partitioned: into journeys through the intersection W of U and V, through U but outside V, and through V outside U. In order to move segments of paths around, by homotopy to form loops returning to a base point w in W, we should assume U, V and W are path-connected; and that W isn't empty. We assume also that U and V are open subspaces with union X.

Under these conditions, $\pi_1(U,w)$, $\pi_1(V,w)$, and $\pi_1(W,w)$, together with the inclusion homomorphisms (induced by the inclusion map):

$$I\colon \pi_1(W,w) \to \pi_1(U,w)$$

and

$$J\colon \pi_1(W,w) \to \pi_1(V,w)$$

are sufficient data to determine $\pi_1(X,w)$. The maps I and J extend to an epimorphism

$$\Phi\colon \pi_1(U,w) * \pi_1(V,w) \to \pi_1(X,w)$$

where $\pi_1(U,w) * \pi_1(V,w)$ is the free product of $\pi_1(U,w)$ and $\pi_1(V,w)$. The kernel of the map Φ are the loops in W that, when viewed in X , are homotopic to the trivial one at w. The group $\pi_1(X,w)$ is therefore isomorphic to $\pi_1(U,w) * \pi_1(V,w)$ modulo such elements, more precisely, to the almagamated free product: $\pi_1(U,w) *_{\pi(W,w)} \pi_1(V,w)$

In particular, when W is simply connected (so that its fundamental group is the trivial group), the theorem says that $\pi_1(X,w)$ is isomorphic to the free product $\pi_1(U,w) * \pi_1(V,w)$.

Equivalent formulations

In the language of combinatorial group theory, $\pi_1(X,w)$ is the free product with amalgamation of those of U and V, with respect to the homomorphisms I and J (which might not be injective): given group presentations

$\pi_1(U,w) = \langle u_1,\ldots,u_k \mid \alpha_1,\ldots,\alpha_l \rangle$

$\pi_1(V,w) = \langle v_1,\ldots,v_m \mid \beta_1,\ldots,\beta_n \rangle$

$\pi_1(W,w) = \langle w_1,\ldots,w_p \mid \gamma_1,\ldots,\gamma_q \rangle$

the amalgamation can be written in terms of generators and relations as $\pi_1(X,w) = \langle u, v \mid \alpha, \beta, \gamma, I(w_r)\cdot J(w_r)^{-1} \rangle$ where each letter u, v, w, α, β, γ stands for the respective set of generators or relators, and the final relator means that the images of each generator w_r under the inclusions I, J are equivalent in the fundamental group of X.

In category theory, the fundamental group of X is a colimit of the diagram of those of U, V and W. More precisely, $\pi_1(X,w)$ is the pushout of the diagram.

Van Kampen's theorem for fundamental groups

Van Kampen's theorem for fundamental groups[1] :

Let X be a topological space which is the union of the interiors of two path connected subspaces $X_1,\ X_2$. Suppose $X_0 := X_1 \cap X_2$ is path connected. Let also $* \in X_0$ and $i_k : \pi_1(X_0, *) \to \pi_1(X_k, *)$, $j_k : \pi_1(X_k, *) \to \pi_1(X, *)$ be induced by the inclusions for k= 1, 2. Then X is path connected and the natural morphism $\pi_1(X_1, *) *_{\pi_1(X_0,*)} \pi_1(X_2, *) \to \pi_1(X, *)$ is an isomorphism, that is, *the fundamental group of X is the free product of the fundamental groups of X_1 and X_2 with amalgamation* of $\pi_1(X_0, *)$.

Usually the morphisms induced by inclusion in this theorem are not themselves injective, and the more precise version of the statement is in terms of pushouts of groups. The notion of pushout in the category of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X, A)$ on a set A of base points,[2] . *This groupoid consists of homotopy classes relative to the end points of paths in X joining points of $A \cap X$. In particular, if X is a contractible space, and A consists of two distinct points of X, then $\pi_1(X, A)$ is easily seen to be isomorphic to the groupoid often written $\mathcal{I}$ with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups*[3] .

Theorem: *Let the topological space X be covered by the interiors of two subspaces X_1, X_2 and let A be a set which meets each path component of X_1, X_2 and $X_0 := X_1 \cap X_2$. Then A meets each path component of X and the diagram **P** of morphisms induced by inclusion*

$$\begin{tikzcd} \pi_1(X_0, A) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1, A) \arrow[d, "\pi_1(j_1)"] \\ \pi_1(X_2, A) \arrow[r, "\pi_1(j_2)"'] & \pi_1(X, A) \end{tikzcd}$$

is a pushout diagram in the category of groupoids.[4]

The interpretation of this theorem as a calculational tool for fundamental groups needs some development of `combinatorial groupoid theory',[5] [6] . This theorem implies the calculation of the fundamental group of the circle as the group of integers, since the group of integers is obtained from the groupoid $\mathcal{I}$ by identifying, in the category of groupoids, its two vertices.

There is a version of the last theorem when X is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets[7] [8] . The conclusion is that if A meets each path component of all 1,2,3-fold intersections of the sets U_λ ,

then A meets all path components of X and the diagram $\bigsqcup_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \rightrightarrows \bigsqcup_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \to \pi_1(X, A)$ of morphi category of groupoids.

Examples

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces. For example, consider the sphere S^2. Pick open sets $A = S^2 - n$ and $B = S^2 - s$ where n and s denote the north and south poles respectively. Then we have the property that A, B and A ⋂ B are open path connected sets. Thus we can see that there is a commutative diagram including A ⋂ B into A and B and then another inclusion from A and B into S^2 and that there is a corresponding diagram of homomorphisms between the fundamental groups of each subspace. Applying Van Kampen's theorem gives the result $\pi_1(S^2) = \pi_1(A) * \pi_1(B)/ker(\Phi)$. However A and B are both homeomorphic to $\mathbf{R^2}$ which is simply connected, so both A and B have trivial fundamental groups. It is clear from this that the fundamental group of S^2 is trivial.

A more complicated example is the calculation of the fundamental group of a genus *n* orientable surface *S*, otherwise known as the *genus n surface group*. One can construct *S* using its standard fundamental polygon. For the first open set *A*, pick a disk within the center of the polygon. Pick *B* to be the complement in *S* of the center point of *A*. Then the intersection of *A* and *B* is an annulus, which is known to be homotopy equivalent to (and so has the same fundamental group as) a circle. Then $\pi_1(A \cap B) = \pi_1(S^1)$, which is the integers, and $\pi_1(A) = \pi_1(D^2) = 1$. Thus the inclusion of $\pi_1(A \cap B)$ into $\pi_1(A)$ sends any generator to the trivial element. However, the inclusion of $\pi_1(A \cap B)$ into $\pi_1(B)$ is not trivial. In order to understand this, first one must calculate $\pi_1(B)$. This is easily done as one can deformation retract B (which is *S* with one point deleted) onto the edges labeled by $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}\ldots A_nB_nA_n^{-1}B_n^{-1}$. This space is known to be the wedge sum of 2*n* circles (also called a bouquet of circles), which further is known to have fundamental group isomorphic to the free group with 2*n* generators, which in this case can be represented by the edges themselves: $\{A_1, B_1, \ldots, A_n, B_n\}$. We now have enough information to apply Van Kampen's theorem. The generators are the loops $\{A_1, B_1, \ldots, A_n, B_n\}$ (*A* is simply connected, so it contributes no generators) and there is exactly one relation: $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}\ldots A_nB_nA_n^{-1}B_n^{-1} = 1$. Using generators and relations, this group is denoted

$$\langle A_1, B_1, \ldots, A_n, B_n | A_1B_1A_1^{-1}B_1^{-1}\ldots A_nB_nA_n^{-1}B_n^{-1}\rangle.$$

Generalizations

This theorem has been extended to the non-connected case by using the fundamental groupoid π_1(X,A) on a *set A of base points*, which consists of homotopy classes of paths in X joining points of X which lie in A. The connectivity conditions for the theorem then become that A meets each path-component of U,V,W. The pushout is now in the category of groupoids. This extended theorem allows the determination of the fundamental group of the circle, and many other useful cases. For example, if the intersection W has two path components, it is convenient to let A consist of one point in each of these components. A theorem for arbitrary covers, with the restriction that A meets all three fold intersections of the sets of the cover, is given in the paper by Brown and Razak cited below. Applications of the fundamental groupoid on a set of base points to the Jordan curve theorem, Covering space, and orbit space are given in Ronald Brown's book cited below.

In the case of orbit spaces, it is convenient to take A to include all the fixed points of the action. An example here is the conjugation action on the circle.

The version that allows more than two overlapping sets but with A a singleton is also given in Allen Hatcher's book below, theorem 1.20.

In fact, we can extend van Kampen's theorem significantly further by considering the → fundamental groupoid $\Pi(X)$, an element of the category of small categories whose objects are points of X and whose arrows are paths between points modulo homotopy equivalence. In this case, to determine the fundamental groupoid of a space, we need only know the fundamental groupoids of the open sets covering X as follows: create a new category in which the objects are fundamental groupoids of the open sets, with an arrow between groupoids if the domain space is a subspace of the codomain. Then van Kampen's theorem is the assertion that the fundamental groupoid of X is the colimit of the diagram. For details, see Peter May's book, chapter 2.

References to higher dimensional versions of the theorem which yield some information on homotopy types are given in an article on higher dimensional group theories and groupoids.[9]

See also

- → Higher dimensional algebra
- → Higher category theory
- → Alexander Grothendieck
- Van Kampen
- → Ronald Brown (mathematician)

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Whitehead's theorem

In → homotopy theory (a branch of mathematics), the **Whitehead theorem** states that if a continuous mapping f between topological spaces X and Y induces isomorphisms on all homotopy groups, then f is a homotopy equivalence provided X and Y are connected CW complexes. This result was proved by → J. H. C. Whitehead in two landmark papers from 1949, and provides a justification for working with the CW complex concept that he introduced there.

Stating it more accurately, we suppose given CW complexes X and Y, with respective base points x and y. Given a continuous mapping

$$f\colon X \to Y$$

such that $f(x) = y$, we consider for $n \geq 0$ the induced homomorphisms

$$f_*\colon \pi_n(X,x) \to \pi_n(Y,y),$$

where π_n denotes for $n \geq 1$ the n-th homotopy group. For $n = 0$ this means the mapping of the path-connected components; if we assume both X and Y are connected we can ignore this as containing no information. We say that f is a **weak homotopy equivalence** if the homomorphisms f_* are all bijective. The Whitehead theorem then states that a weak homotopy equivalence, for connected CW complexes, is an actual homotopy equivalence.

A word of caution: it is not enough to assume $\pi_n(X)$ is isomorphic to $\pi_n(Y)$ for each $n \geq 1$ in order to conclude that X and Y are homotopy equivalent. One really needs a map $f : X \to Y$ inducing such isomorphisms in homotopy. For instance, take $X = S^2 \times \mathbf{RP}^3$ and $Y = \mathbf{RP}^2 \times S^3$. Then X and Y have the same fundamental group, namely $\mathbf{Z}_2$, and the same universal cover, namely $S^2 \times S^3$; thus, they have isomorphic homotopy groups. On the other hand their homology groups are different (as can be seen from the Künneth formula); thus, X and Y are not homotopy equivalent.

The Whitehead theorem does not hold for general topological spaces or even for all subspaces of $\mathbf{R^n}$. For example, the Warsaw circle, a subset of the plane, has all homotopy groups zero, but the map from the Warsaw circle to a single point is not a homotopy equivalence. The study of possible generalizations of Whitehead's theorem to more general spaces is part of the subject of shape theory.

Generalization to model categories

In any model category, a weak equivalence between cofibrant-fibrant objects is a homotopy equivalence.

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Important publications in algebraic topology

List of publications in mathematics

This is a list of **important publications** in mathematics, organized by field.

Some reasons why a particular publication might be regarded as important:

- **Topic creator** – A publication that created a new topic
- **Breakthrough** – A publication that changed scientific knowledge significantly
- **Introduction** – A publication that is a good introduction or survey of a topic
- **Influence** – A publication which has significantly influenced the world
- **Latest and greatest** – The current most advanced result in a topic

Algebra

Theory of equations

Hisab al-Jabr w'al-muqabala, Kitab al-Jabr wa-l-Muqabala

- Muhammad ibn Mūsā al-Khwārizmī (820)

Description: The first book on the systematic algebraic solutions of linear and quadratic equations. The book is considered to be the foundation of modern algebra and Islamic mathematics. The word "algebra" itself is derived from the *al-Jabr* in the title of the book.

Ars Magna

- Gerolamo Cardano (1545)

Description: Provided the first published methods for solving cubic and quartic equations (due to Scipione del Ferro, Niccolò Fontana Tartaglia, and Lodovico Ferrari), and exhibited the first published calculations involving non-real complex numbers. [1]

Vollständige Anleitung zur Algebra

- Leonhard Euler (1770)

Description: Also known as Elements of Algebra, Euler's textbook on elementary algebra is one of the first to set out algebra in the modern form we would recognize today. The first volume deals with determinate equations, while the second part deals with Diophantine equations. The last section contains a proof of Fermat's Last Theorem for the case $n = 3$, making some valid assumptions regarding $Q(\sqrt{-3})$ that Euler did not prove.[2]

Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse

- Carl Friedrich Gauss (1799)

Description: Gauss' doctoral dissertation,[3] which contained a widely accepted (at the time) but incomplete proof[4] of the fundamental theorem of algebra.

Abstract algebra

Group theory

Réflexions sur la résolution algébrique des équations

- Joseph Louis Lagrange (1770)

Description: Made the prescient observation that the roots of the Lagrange resolvent of a polynomial equation are tied to permutations of the roots of the original equation, laying a more general foundation for what had previously been an ad hoc analysis and helping motivate the later development of the theory of permutation groups, group theory, and Galois theory. The Lagrange resolvent also introduced the discrete Fourier transform of order 3.

Articles Publiés par Galois dans les Annales de Mathématiques

- Journal de Mathematiques pures et Appliquées, II (1846)

Description: Posthumous publication of the mathematical manuscripts of Évariste Galois by Joseph Liouville. Included are Galois' papers *Mémoire sur les conditions de résolubilité des équations par radicaux* and *Des équations primitives qui sont solubles par radicaux*.

Traité des substitutions et des équations algébriques

- Camille Jordan (1870)

Description: The first book on group theory, giving a then-comprehensive study of permutation groups and Galois theory. In this book, Jordan introduced the notion of a simple group and epimorphism (which he called *l'isomorphisme mériédrique*),[5] proved part of the Jordan–Hölder theorem, and discussed matrix groups over finite fields as well as the Jordan normal form. [6]

Theorie der Transformationsgruppen

- Sophus Lie, Friedrich Engel (1888-1893).

Publication data: 3 volumes, B.G. Teubner, Verlagsgesellschaft, mbH, Leipzig, 1888-1893. Volume 1 [7], Volume 2 [8], Volume 3 [8].

Description: The first comprehensive work on transformation groups, serving as the foundation for the modern theory of Lie groups.

Solvability of groups of odd order

- Walter Feit and John Thompson (1960)

Description: Gave a complete proof of the solvability of finite groups of odd order, establishing the long-standing Burnside conjecture that all finite non-abelian simple groups are of even order. Many of the original techniques used in this paper were used in the eventual classification of finite simple groups.

→ Homological algebra

Homological Algebra

- → Henri Cartan and → Samuel Eilenberg (1956)

Description: Provided the first fully-worked out treatment of abstract homological algebra, unifying previously disparate presentations of homology and cohomology for associative algebras, Lie algebras, and groups into a single theory.

Sur Quelques Points d'Algèbre Homologique

- → Alexander Grothendieck (1957)

Description: Revolutionized → homological algebra by introducing abelian categories and providing a general framework for Cartan and Eilenberg's notion of derived functors.

Algebraic geometry

Theorie der Abelschen Functionen

- Bernhard Riemann (1857)

Publication data: *Journal für die Reine und Angewandte Mathematik*

Description: Developed the concept of Riemann surfaces and their topological properties beyond Riemann's 1851 thesis work, proved an index theorem for the genus (the original formulation of the Riemann-Hurwitz formula), proved the Riemann inequality for the dimension of the space of meromorphic functions with prescribed poles (the original formulation of the Riemann-Roch theorem), discussed birational transformations of a given curve and the dimension of the corresponding moduli space of inequivalent curves of a given genus, and solved more general inversion problems than those investigated by Abel and Jacobi. André Weil once wrote that this paper "*is one of the greatest pieces of mathematics that has ever been written; there is not a single word in it that is not of consequence.*" [9]

Faisceaux Algébriques Cohérents

- → Jean-Pierre Serre

Publication data: *Annals of Mathematics*, 1955

Description: *FAC*, as it is usually called, first introduced the use of sheaves into algebraic geometry. Serre introduced Čech cohomology of sheaves in this paper, and, despite its technical deficiencies, revolutionized algebraic geometry. For example, the long exact sequence in sheaf cohomology allows one to show that some surjective maps of sheaves induce surjective maps on sections; specifically, these are the maps whose kernel (as a sheaf) has a vanishing first cohomology group. Before *FAC*, this was next to impossible. While Grothendieck's derived functor cohomology has replaced Čech cohomology for technical reasons, actual calculations, such as of the cohomology of projective space, are usually carried out by Čech techniques, and for this reason Serre's paper remains important even today.

Géométrie Algébrique et Géométrie Analytique

- → Jean-Pierre Serre (1956)

Description: In mathematics, algebraic geometry and analytic geometry are closely related subjects, where *analytic geometry* is the theory of complex manifolds and the more general analytic spaces defined locally by the vanishing of analytic functions of several complex variables. A (mathematical) theory of the relationship between the two was put in place during the early part of the 1950s, as part of the business of laying the foundations of algebraic geometry to include, for example, techniques from Hodge theory. (*NB* While analytic geometry as use of Cartesian coordinates is also in a sense included in the scope of algebraic geometry, that is not the topic being discussed in this article.) The major paper consolidating the theory was *Géometrie Algébrique et Géométrie Analytique* by → Serre, now usually referred to as *GAGA*. A *GAGA-style result* would now mean any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, and a well-defined subcategory of analytic geometry objects and holomorphic mappings.

Le théorème de Riemann-Roch, d'après A. Grothendieck

- Armand Borel, → Jean-Pierre Serre (1958)

Description: Borel and Serre's exposition of Grothendieck's version of the Riemann Roch theorem, published after Grothendieck made it clear that he was not interested in writing up his own result. Grothendieck reinterpreted both sides of the formula that Hirzebruch proved in 1953 in the framework of morphisms between varieties, resulting in a sweeping generalization.[10] In his proof, Grothendieck broke new ground with his concept of Grothendieck groups, which led to the development of → K-theory.[11]

Éléments de géométrie algébrique

- → Alexander Grothendieck (1960-1967)

Description: Written with the assistance of Jean Dieudonné, this is Grothendieck's exposition of his reworking of the foundations of algebraic geometry. It has become the most important foundational work in modern algebraic geometry. The approach expounded in EGA, as these books are known, transformed the field and led to monumental advances.

Séminaire de géométrie algébrique

- → Alexander Grothendieck et al.

Description: These seminar notes on Grothendieck's reworking of the foundations of algebraic geometry report on work done at IHÉS starting in the 1960s. SGA 1 dates from the seminars of 1960-1961, and the last in the series, SGA 7, dates from 1967–1969. In contrast to EGA, which is intended to set foundations, SGA describes ongoing research as it unfolded in Grothendieck’s seminar; as a result, it is quite difficult to read, since many of the more elementary and foundational results were relegated to EGA. One of the major results building on the results in SGA is Pierre Deligne's proof of the last of the open Weil conjectures in the early 1970s. Other authors who worked on one or several volumes of SGA include Michel Raynaud, Michael Artin, → Jean-Pierre Serre, Jean-Louis Verdier, Pierre Deligne, and Nicholas Katz.

Number theory

De fractionibus continuis dissertatio

- Leonhard Euler (1744)

Description: First presented in 1737, this paper [12] provided the first then-comprehensive account of the properties of continued fractions. It also contains the first proof that the number e is irrational.[13]

Recherches d'Arithmétique

- Joseph Louis Lagrange (1775)

Description: Developed a general theory of binary quadratic forms to handle the general problem of when an integer is representable by the form $ax^2 + by^2 + cxy$. This included a reduction theory for binary quadratic forms, where he proved that every form is equivalent to a certain canonically chosen reduced form.[14] [15]

Disquisitiones Arithmeticae

- Carl Friedrich Gauss (1801)

Description: The *Disquisitiones Arithmeticae* is a profound and masterful book on number theory written by German mathematician Carl Friedrich Gauss and first published in 1801 when Gauss was 24. In this book Gauss brings together results in number theory obtained by mathematicians such as Fermat, Euler, Lagrange and Legendre and adds many important new results of his own. Among his contributions was the first complete proof known of the Fundamental theorem of arithmetic, the first two published proofs of the law of quadratic reciprocity, a deep investigation of binary quadratic forms going beyond Lagrange's work in Recherches d'Arithmétique, a first appearance of Gauss sums, cyclotomy, and the theory of constructible polygons with a particular application to the constructibility of the regular 17-gon. Of note, in section V, article 303 of Disquisitiones, Gauss summarized his calculations of class numbers of imaginary quadratic number fields, and in fact found all imaginary quadratic number fields of class numbers 1, 2, and 3 (confirmed in 1986) as he had conjectured.[16] In section V, article 358, Gauss proved what can be interpreted as the first non-trivial case of the Riemann Hypothesis for curves over finite fields (the Hasse-Weil theorem).[17]

Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthä lt

- Johann Peter Gustav Lejeune Dirichlet (1837)

Description: Pioneering paper in analytic number theory, which introduced Dirichlet characters and their L-functions to establish Dirichlet's theorem on arithmetic progressions.[18] In subsequent publications, Dirichlet used these tools to determine, among other things, the class number for quadratic forms.

Über die Anzahl der Primzahlen unter einer gegebenen Grösse

- Bernhard Riemann (1859)

Description: *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (or *On the Number of Primes Less Than a Given Magnitude*) is a seminal 8-page paper by Bernhard Riemann published in the November 1859 edition of the *Monthly Reports of the Berlin Academy*. Although it is the only paper he ever published on number theory, it contains ideas which influenced dozens of researchers during the late 19th century and up to the present day. The paper consists primarily of definitions, heuristic arguments, sketches of proofs, and the application of powerful analytic methods; all of these have become essential concepts and tools of modern analytic number theory. It also contains the famous Riemann Hypothesis, one of the most important open problems in mathematics.

Vorlesungen über Zahlentheorie

- P.G.L. Dirichlet and Richard Dedekind

Description: *Vorlesungen über Zahlentheorie* (*Lectures on Number Theory*) is a textbook of number theory written by German mathematicians P.G.L. Dirichlet and Richard Dedekind, and published in 1863. The *Vorlesungen* can be seen as a watershed between the classical number theory of Fermat, Jacobi and Gauss, and the modern number theory of Dedekind, Riemann and Hilbert. Dirichlet does not explicitly recognise the concept of the group that is central to modern algebra, but many of his proofs show an implicit understanding of group theory

Zahlbericht

- David Hilbert (1897)

Description: Unified and made accessible many of the developments in algebraic number theory made during the nineteenth century. Although criticized by André Weil (who stated "*more than half of his famous Zahlbericht is little more than an account of Kummer's number-theoretical work, with inessential improvements*")[19] and Emmy Noether,[20] it was highly influential for many years following its publication.

Fourier Analysis in Number Fields and Hecke's Zeta-Functions

- John Tate (1950)

Description: Generally referred to simply as *Tate's Thesis*, Tate's Princeton Ph.D. thesis, under Emil Artin, is a reworking of Erich Hecke's theory of zeta- and *L*-functions in terms of Fourier analysis on the adeles. The introduction of these methods into number theory made it possible to formulate extensions of Hecke's results to more general *L*-functions such as those arising from automorphic forms.

Automorphic Forms on GL(2)

- Hervé Jacquet and Robert Langlands (1970)

Description: This publication offers evidence towards Langlands' conjectures by reworking and expanding the classical theory of modular forms and their *L*-functions through the introduction of representation theory.

La conjecture de Weil. I.

- Pierre Deligne (1974)

Description: Proved the Riemann hypothesis for varieties over finite fields, settling the last of the open Weil conjectures.

Endlichkeitssätze für abelsche Varietäten über Zahlkörpern

- Gerd Faltings (1983)

Description: Faltings proves a collection of important results in this paper, the most famous of which is the first proof of the Mordell conjecture (a conjecture dating back to 1922). Other theorems proved in this paper include an instance of the Tate conjecture (relating the homomorphisms between two abelian varieties over a number field to the homomorphisms between their Tate modules) and some finiteness results concerning abelian varieties over number fields with certain properties.

Modular Elliptic Curves and Fermat's Last Theorem

- Andrew Wiles (1995)

Description: This article proceeds to prove a special case of the Shimura-Taniyama conjecture through the study of the deformation theory of Galois representations. This in turn implies the famed Fermat's Last Theorem. The proof's method of identification of a deformation ring with a Hecke algebra (now referred to as an $R=T$ theorem) to prove modularity lifting theorems has been an influential development in algebraic number theory.

The geometry and cohomology of some simple Shimura varieties

- Michael Harris and Richard Taylor (2001)

Description: Harris and Taylor provide the first proof of the local Langlands conjecture for GL(n). As part of the proof, this monograph also makes an in depth study of the geometry and cohomology of certain Shimura varieties at primes of bad reduction.

Analysis

Introductio in analysin infinitorum

- Leonhard Euler (1748)

Description: The eminent historian of mathematics Carl Boyer once called Euler's *Introductio in analysin infinitorum* the greatest modern textbook in mathematics.[21] Published in two volumes,[22] [23] this book more than any other work succeeded in establishing analysis as a major branch of mathematics, with a focus and approach distinct from that used in geometry and algebra.[24] Notably, Euler identified functions rather than curves to be the central focus in his book.[25] Logarithmic, exponential, trigonometric, and transcendental functions were covered, as were expansions into partial fractions, evaluations of ζ(2k) for k a positive integer between 1 and 13, infinite series-infinite product formulas,[21] continued fractions, and partitions of integers.[26] In this work, Euler proved that every rational number can be written as a finite continued fraction, that the continued fraction of an irrational number is infinite, and derived continued fraction expansions for e and $\sqrt{e}$.[22] This work also contains a statement of Euler's formula and a statement of the pentagonal number theorem, which he had discovered earlier and would publish a proof for in 1751.

Calculus

Yuktibhasa

- Jyeshtadeva (1501)

Description: Written in India in 1501, this was the world's first calculus text. "This work laid the foundation for a complete system of fluxions" (Charles Whish, 1835) and served as a summary of the Kerala School's achievements in calculus, trigonometry and mathematical analysis, most of which were earlier discovered by the 14th century mathematician Madhava. It's possible that this text influenced the later development of calculus in Europe. Some of its important developments in calculus include: the fundamental ideas of differentiation and integration, the derivative, differential equations, term by term integration, numerical integration by means of infinite series, the relationship between the area of a curve and its integral, and the mean value theorem.

Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus

- Gottfried Leibniz (1684)

Description: Leibniz's first publication on differential calculus, containing the now familiar notation for differentials as well as rules for computing the derivatives of powers, products and quotients.

Philosophiae Naturalis Principia Mathematica

- Isaac Newton

Description: The ***Philosophiae Naturalis Principia Mathematica*** (Latin: "mathematical principles of natural philosophy", often *Principia* or *Principia Mathematica* for short) is a three-volume work by Isaac Newton published on July 5, 1687. Perhaps the most influential scientific book ever published, it contains the statement of Newton's laws of motion forming the foundation of classical mechanics as well as his law of universal gravitation, and derives Kepler's laws for the motion of the planets (which were first obtained empirically). Here was born the practice, now so standard we identify it with science, of explaining nature by postulating mathematical axioms and demonstrating that their conclusion are observable phenomena. In formulating his physical theories, Newton freely used his unpublished work on calculus. When he submitted Principia for publication, however, Newton chose to recast the majority of his proofs as geometric arguments.[27]

Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum

- Leonhard Euler (1755)

Description: Published in two books,[28] Euler's textbook on differential calculus presented the subject in terms of the function concept, which he had introduced in his 1748 *Introductio in analysin infinitorum*. This work opens with a study of the calculus of finite differences and makes a thorough investigation of how differentiation behaves under substitutions.[1] Also included is a systematic study of Bernoulli polynomials and the Bernoulli numbers (naming them as such), a demonstration of how the Bernoulli numbers are related to the coefficients in the Euler–Maclaurin formula and the values of $\zeta(2n)$,[29] a further study of Euler's constant (including its connection to the gamma function), and an application of partial fractions to differentiation. [30]

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

- Bernhard Riemann (1867)

Description: Written in 1853, Riemann's work on trigonometric series was published posthumously. In it, he extended Cauchy's definition of the integral to that of the Riemann integral, allowing some functions with dense subsets of discontinuities on an interval to be integrated (which he demonstrated by an example).[31] He also stated the Riemann series theorem,[31] proved the Riemann-Lebesgue lemma for the case of bounded Riemann integrable functions,[32] and developed the Riemann localization principle.[33]

Intégrale, longueur, aire

- Henri Lebesgue (1901)

Description: Lebesgue's doctoral dissertation, summarizing and extending his research to date regarding his development of measure theory and the Lebesgue integral.

Complex analysis

Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse

- Bernhard Riemann (1851)

Description: Riemann's doctoral dissertation introduced the notion of a Riemann surface, conformal mapping, simple connectivity, the Riemann sphere, the Laurent series expansion for functions having poles and branch points, and the Riemann mapping theorem.

Functional analysis

Théorie des opérations linéaires

- Stefan Banach (1932; originally published 1931 in Polish under the title *Teorja operacyj*.)

Description: The first mathematical monograph on the subject of linear metric spaces, bringing the abstract study of functional analysis to the wider mathematical community. The book introduced the ideas of a normed space and the notion of a so-called *B*-space, a complete normed space. The *B*-spaces are now called Banach spaces and are one of the basic objects of study in all areas of modern mathematical analysis. Banach also gave proofs of versions of the open mapping theorem, closed graph theorem, and Hahn-Banach theorem.

Fourier analysis

Mémoire sur la propagation de la chaleur dans les corps solides

- Joseph Fourier (1807)[34]

Description: Introduced Fourier analysis, specifically Fourier series. Key contribution was to not simply use trigonometric series, but to model *all* functions by trigonometric series.

> “$\varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \cdots.$”
>
> Multiplying both sides by $\cos(2i+1)\frac{\pi y}{2}$, and then integrating from $y=-1$ to $y=+1$ yields:
>
> $a_i = \int_{-1}^{1} \varphi(y)\cos(2i+1)\frac{\pi y}{2}\,dy.$

When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: ...*the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour*. Making Fourier series rigorous, which in detail took over a century, led directly to a number of developments in analysis, notably the rigorous statement of the integral via the Dirichlet integral and later the Lebesgue integral.

Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données

- Johann Peter Gustav Lejeune Dirichlet (1829, expanded German edition in 1837)

Description: In his habilitation thesis on Fourier series, Riemann characterized this work of Dirichlet as the "*the first profound paper about the subject*".[35] This paper gave the first rigorous proof of the convergence of Fourier series under fairly general conditions (piecewise continuity and monotonicity) by considering partial sums, which Dirichlet transformed into a particular Dirichlet integral involving what is now called the Dirichlet kernel. This paper introduced the nowhere continuous Dirichlet function and an early version of the Riemann-Lebesgue lemma.[36]

On convergence and growth of partial sums of Fourier series

- Lennart Carleson (1966)

Description: Settled Lusin's conjecture that the Fourier expansion of any L^2 function converges almost everywhere.

Geometry

Baudhayana Sulba Sutra

- Baudhayana

Description: Written around the 8th century BC, this is one of the oldest geometrical texts. It laid the foundations of Indian mathematics and was influential in South Asia and its surrounding regions, and perhaps even Greece. Among the important geometrical discoveries included in this text are: the earliest list of Pythagorean triples discovered algebraically, the earliest statement of the Pythagorean theorem, geometric solutions of linear equations, several approximations of π, the first use of irrational numbers, and an accurate computation of the square root of 2, correct to a remarkable five decimal places. Though this was primarily a geometrical text, it also contained some important algebraic developments, including the earliest use of quadratic equations of the forms $ax^2 = c$ and $ax^2 + bx = c$, and integral solutions of simultaneous Diophantine equations with up to four unknowns.

Euclid's Elements

- Euclid

Publication data: c. 300 BC

Online version: Interactive Java version [37]

Description: This is often regarded as not only the most important work in geometry but one of the most important works in mathematics. It contains many important results in geometry, number theory and the first algorithm as well. More than any specific result in the publication, it seems that the major achievement of this publication is the popularization of logic and mathematical proof as a method of solving problems.

The Nine Chapters on the Mathematical Art

- Unknown author

Description: This was a Chinese mathematics book, mostly geometric, composed during the Han Dynasty, perhaps as early as 200 BC. It remained the most important textbook in China and East Asia for over a thousand years, similar to the position of Euclid's *Elements* in Europe. Among its contents: Linear problems solved using the principle known later in the West as the *rule of false position*. Problems with several unknowns, solved by a principle similar to Gaussian elimination. Problems involving the principle known in the West as the Pythagorean theorem. The earliest solution of a matrix using a method equivalent to the modern method.

The Conics

- Apollonius of Perga

Description: The Conics was written by Apollonius of Perga, a Greek mathematician. His innovative methodology and terminology, especially in the field of conics, influenced many later scholars including Ptolemy, Francesco Maurolico, Isaac Newton, and René Descartes. It was Apollonius who gave the ellipse, the parabola, and the hyperbola the names by which we know them.

La Géométrie

- René Descartes

Description: La Géométrie was published in 1637 and written by René Descartes. The book was influential in developing the Cartesian coordinate system and specifically discussed the representation of points of a plane, via real numbers; and the representation of curves, via equations.

Grundlagen der Geometrie

- David Hilbert

Publication data: Hilbert, David (1899). *Grundlagen der Geometrie*. Teubner-Verlag Leipzig.

Description: Hilbert's axiomatization of geometry, whose primary influence was in its pioneering approach to metamathematical questions including the use of models to prove axiom independence and the importance of establishing the consistency and completeness of an axiomatic system.

Regular Polytopes

- H.S.M. Coxeter

Description: *Regular Polytopes* is a comprehensive survey of the geometry of regular polytopes, the generalisation of regular polygons and regular polyhedra to higher dimensions. Originating with an essay entitled *Dimensional Analogy* written in 1923, the first edition of the book took Coxeter 24 years to complete. Originally written in 1947, the book was updated and republished in 1963 and 1973.

Differential geometry

Recherches sur la courbure des surfaces

- Leonard Euler (1760)

Publication data: Mémoires de l'académie des sciences de Berlin **16** (1760) pp. 119–143; published 1767. (Full text [38] and an English translation available from the Dartmouth Euler archive.)

Description: Established the theory of surfaces, and introduced the idea of principal curvatures, laying the foundation for subsequent developments in the differential geometry of surfaces.

Disquisitiones generales circa superficies curvas

- Carl Friedrich Gauss (1827)

Publication data: "Disquisitiones generales circa superficies curvas" [39], *Commentationes Societatis Regiae Scientiarum Gottingesis Recentiores* Vol. **VI** (1827), pp. 99–146; "General Investigations of Curved Surfaces [40]" (published 1965) Raven Press, New York, translated by A.M.Hiltebeitel and J.C.Morehead.

Description: Groundbreaking work in differential geometry, introducing the notion of Gaussian curvature and Gauss' celebrated Theorema Egregium.

Über die Hypothesen, welche der Geometrie zu Grunde Liegen

- Bernhard Riemann (1854)

Publication data: "Über die Hypothesen, welche der Geometrie zu Grunde Liegen" [41], *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 13, 1867.

Description: Riemann's famous Habiltationsvortrag, in which he introduced the notions of a manifold, Riemannian metric, and curvature tensor.

Leçons sur la théorie génerale des surfaces

- Gaston Darboux

Publication data: Darboux, Gaston (1887,1889,1896). *Leçons sur la théorie génerale des surfaces: Volume I* [42], *Volume II* [43], *Volume III* [44], *Volume IV* [44]. Gauthier-Villars.

Description: A treatise covering virtually every aspect of the 19th century differential geometry of surfaces.

→ Topology

Analysis situs

- Henri Poincaré (1895, 1899-1905)

Description: Poincaré's Analysis situs and his Compléments à l'Analysis Situs laid the general foundations for → algebraic topology. In these papers, Poincaré introduced the notions of homology and the → fundamental group, provided an early formulation of → Poincaré duality, gave the Euler-Poincaré characteristic for chain complexes, and mentioned several important conjectures including the Poincaré conjecture.

Grundzüge der Mengenlehre

- Felix Hausdorff (1914)
- Reprinted and commented in: *Gesammelte Werke Bd. 2* / Herausgegeben von E. Brieskorn, S.D.Chatterji *et al.*. – Berlin, 2002

Description: This book founded (general) topology by giving the axioms for a (Hausdorff) topological space.

L'anneau d'homologie d'une représentation, *Structure de l'anneau d'homologie d'une représentation*

- Jean Leray (1946)

Description: These two Comptes Rendus notes of Leray from 1946 introduced the novel concepts of sheafs, sheaf cohomology, and spectral sequences, which he had developed during his years of captivity as a prisoner of war. Leray's announcements and applications (published in other Comptes Rendus notes from 1946) drew immediate attention from other mathematicians. Subsequent clarification, development, and generalization by → Henri Cartan, Jean-Louis Koszul, Armand Borel, → Jean-Pierre Serre, and Leray himself allowed these concepts to be understood

and applied to many other areas of mathematics.[45] Dieudonné would later write that these notions created by Leray "*undoubtedly rank at the same level in the history of mathematics as the methods invented by Poincaré and Brouwer*". [11]

Quelques propriétés globales des variétés differentiables

- → René Thom (1954)

Description: In this paper, Thom proved the Thom transversality theorem, introduced the notions of oriented and unoriented cobordism, and demonstrated that cobordism groups could be computed as the homotopy groups of certain Thom spaces. Thom completely characterized the unoriented cobordism ring and achieved strong results for several problems, including Steenrod's problem on the realization of cycles.[11] [46]

Category theory

General theory of natural equivalences

- → Samuel Eilenberg and → Saunders Mac Lane (1945)

Description: The first paper on category theory. Mac Lane later wrote in *Categories for the Working Mathematician* that he and Eilenberg introduced categories so that they could introduce functors, and they introduced functors so that they could introduce natural equivalences. Prior to this paper, "natural" was used in an informal and imprecise way to designate constructions that could be made without making any choices. Afterwards, "natural" had a precise meaning which occurred in a wide variety of contexts and had powerful and important consequences.

Categories for the Working Mathematician

- → Saunders Mac Lane (1971, second edition 1998)

Description: Saunders Mac Lane, one of the founders of category theory, wrote this exposition to bring categories to the masses. Mac Lane does not get lost in pointless abstraction, but instead brings to the fore the important concepts that make category theory useful, such as adjoint functors and universal properties. His text is more comprehensive than most mathematicians will ever need, and consequently is also an excellent reference.

Set theory

Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen

- Georg Cantor (1874)

Description: Contains the first proof that the set of all real numbers is uncountable; also contains a proof that the set of algebraic numbers is denumerable.

Grundzüge der Mengenlehre

- Felix Hausdorff

Description: First published in 1914, this was the first comprehensive introduction to set theory. Besides the systematic treatment of known results in set theory, the book also contains chapters on measure theory and topology, which were then still considered parts of set theory. Here Hausdorff presents and develops highly original material which was later to become the basis for those areas.

The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory

- Kurt Gödel (1938)

Description: Gödel proves the results of the title. Also, in the process, introduces the class L of constructible sets, a major influence in the development of axiomatic set theory.

The Independence of the Continuum Hypothesis

- Paul J. Cohen (1963, 1964)

Description: Cohen's breakthrough work proved the independence of the continuum hypothesis and axiom of choice with respect to Zermelo-Fraenkel set theory. In proving this Cohen introduced the concept of *forcing* which led to many other major results in axiomatic set theory.

Logic

Begriffsschrift

- Gottlob Frege (1879)

Description: Published in 1879, the title ***Begriffsschrift*** is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "*a formula language, modelled on that of arithmetic, of pure thought*". Frege's motivation for developing his formal logical system was similar to Leibniz's desire for a *calculus ratiocinator*. Frege defines a logical calculus to support his research in the foundations of mathematics. ***Begriffsschrift*** is both the name of the book and the calculus defined therein. It was arguably the most significant publication in logic since Aristotle.

Formulario mathematico

- Giuseppe Peano (1895)

Description: First published in 1895, the **Formulario mathematico** was the first mathematical book written entirely in a formalized language. It contained a description of mathematical logic and many important theorems in other branches of mathematics. Many of the notations introduced in the book are now in common use.

Principia Mathematica

- Bertrand Russell and Alfred North Whitehead (1910-1913)

Description: The ***Principia Mathematica*** is a three-volume work on the foundations of mathematics, written by Bertrand Russell and Alfred North Whitehead and published in 1910-1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The questions remained whether a contradiction could be derived from the Principia's axioms, and whether there exists a mathematical statement which could neither be proven nor disproven in the system. These questions were settled, in a rather surprising way, by Gödel's incompleteness theorem in 1931.

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I

- Kurt Gödel (1931)

Online version: Online version [47]

Description: In mathematical logic, **Gödel's incompleteness theorems** are two celebrated theorems proved by Kurt Gödel in 1931. The first incompleteness theorem states:

> For any formal system such that (1) it is ω-consistent (omega-consistent), (2) it has a recursively definable set of axioms and rules of derivation, and (3) every recursive relation of natural numbers is definable in it, there exists a formula of the system such that, according to the intended interpretation of the system, it expresses a truth about natural numbers and yet it is not a theorem of the system.

Combinatorics

On sets of integers containing no k elements in arithmetic progression

- Endre Szemerédi (1975)

Description: Settled a conjecture of Paul Erdős and Paul Turán that if a sequence of natural numbers has positive upper density then it contains arbitrarily long arithmetic progressions. Szemerédi's solution has been described as a "masterpiece of combinatorics"[48] and it introduced new ideas and tools to the field including the Szemerédi regularity lemma.

Graph theory

Solutio problematis ad geometriam situs pertinentis

- Leonhard Euler (1741)
- Euler's original publication [49] (in Latin)

Description: Euler's solution of the Königsberg bridge problem in *Solutio problematis ad geometriam situs pertinentis* (*The solution of a problem relating to the geometry of position*) is considered to be the first theorem of graph theory.

On the evolution of random graphs

- Paul Erdös and Alfréd Rényi (1960)

Description: Provides a detailed discussion of sparse random graphs, including distribution of components, occurrence of small subgraphs, and phase transitions.[50]

Network Flows and General Matchings

- Ford, L., & Fulkerson, D.
- Flows in Networks. Prentice-Hall, 1962.

Description: Ford and Fulkerson paper on Network Flows. The algorithm along with many ideas on flow-based models can be found in their book.

Complexity theory

See List of important publications in computer science.

Probability theory

See list of important publications in statistics.

Game theory

Zur Theorie der Gesellschaftsspiele

- John von Neumann (1928)

Description: Went well beyond Émile Borel's initial investigations into strategic two-person game theory by proving the minimax theorem for two-person, zero-sum games.

Theory of Games and Economic Behavior

- Oskar Morgenstern, John von Neumann (1944)

Description: This book led to the investigation of modern game theory as a prominent branch of mathematics. This profound work contained the method for finding optimal solutions for two-person zero-sum games.

Equilibrium Points in N-person Games

- John Forbes Nash
- *Proceedings of the National Academy of Sciences* 36 (1950), 48–49. MR0031701 [51]
- "Equilibrium Points in N-person Games" [52]

Description: Nash equilibrium

On Numbers and Games

- John Horton Conway

Description: The book is in two, {0,1|}, parts. The zeroth part is about numbers, the first part about games - both the values of games and also some real games that can be played such as Nim, Hackenbush, Col and Snort amongst the many described.

Winning Ways for your Mathematical Plays

- Elwyn Berlekamp, John Conway and Richard K. Guy

Description: A compendium of information on mathematical games. It was first published in 1982 in two volumes, one focusing on Combinatorial game theory and surreal numbers, and the other concentrating on a number of specific games.

Fractals

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

- Benoît Mandelbrot

Description: A discussion of self-similar curves that have fractional dimensions between 1 and 2. These curves are examples of fractals, although Mandelbrot does not use this term in the paper, as he did not coin it until 1975. Shows Mandelbrot's early thinking on fractals, and is an example of the linking of mathematical objects with natural forms that was a theme of much of his later work.

Numerical analysis

Numerical linear algebra

The Algebraic Eigenvalue Problem

- James H. Wilkinson (1965)

Description:

Optimization

Method of Fluxions

- Isaac Newton

Description: *Method of Fluxions* was a book written by Isaac Newton. The book was completed in 1671, and published in 1736. Within this book, Newton describes a method (the Newton-Raphson method) for finding the real zeroes of a function.

Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies

- Joseph Louis Lagrange (1761)

Description: Major early work on the calculus of variations, building upon some of Lagrange's prior investigations as well as those of Euler. Contains investigations of minimal surface determination as well as the initial appearance of Lagrange multipliers.

The New Variational Method

- Leonid Kantorovich
- This work was published in 1930's in the erstwhile Soviet Republic.

Description: Kantorovich wrote the first paper on production planning, which used Linear Programs as the model. He proposed the simplex algorithm as a systematic procedure to solve these Linear Programs. He received the Nobel prize for this work in 1975.

Decomposition Principle for Linear Programs

- George Dantzig and P. Wolfe
- Operations Research 8:101–111, 1960.

Description: Dantzig's is considered the father of Linear Programming in the western world. He independently invented the simplex algorithm. Dantzig and Wolfe worked on decomposition algorithms for large scale linear programs in factory and production planning.

How good is the simplex algorithm?

- Victor Klee and George J. Minty
- In: O. Shisha (ed.) Inequalities III, Academic Press (1972) 159–175.

Description: Klee and Minty gave an example showing that the simplex method can take exponentially many steps to solve a linear program if it chooses the greedy ascent rule.

Linear Programming and Polynomial time algorithms

- L. Khachiyan
- Doklady Akademii Nauk SSSR 244 (1979) pp. 1093–1096 (Russian).

Description: Khachiyan's work on Ellipsoid method. This was the first polynomial time algorithm for Linear programming.

New polynomial-time algorithm for linear programming

- Karmarkar, N.
- Combinatorica 4, 373–395, 1984.

Description: Karmarkars path-breaking work on Interior-Point algorithms for Linear Programming.

Interior Point Polynomial Algorithms in Convex Programming

- Yurii Nesterov and A. Nemirovski
- Philadelphia : Society for Industrial and Applied Mathematics, 1994. (SIAM Studies in Applied Mathematics).

Description: Nesterov and Nemirovski's work on self-concordant barriers and interior-point methods for general convex programming. All their series of papers (both individual and combined) is compiled more coherently in the following "bible" of convex optimization.

Early manuscripts

These are publications that are not necessarily relevant to a mathematician nowadays, but are nonetheless important publications in the history of mathematics.

Rhind Mathematical Papyrus

- Ahmes (scribe)

Description: It is one of the oldest mathematical texts, dating to the Second Intermediate Period of ancient Egypt. It was copied by the scribe Ahmes (properly *Ahmose*) from an older Middle Kingdom papyrus. It laid the foundations of Egyptian mathematics and in turn, later influenced Greek and Hellenistic mathematics. Besides describing how to obtain an approximation of π only missing the mark by less than one per cent, it is describes one of the earliest attempts at squaring the circle and in the process provides persuasive evidence against the theory that the Egyptians deliberately built their pyramids to enshrine the value of π in the proportions. Even though it would be a strong overstatement to suggest that the papyrus represents even rudimentary attempts at analytical geometry, Ahmes did make use of a kind of an analogue of the cotangent.

Archimedes Palimpsest

- Archimedes of Syracuse

Description: Although the only mathematical tools at its author's disposal were what we might now consider secondary-school geometry, he used those methods with rare brilliance, explicitly using infinitesimals to solve problems that would now be treated by integral calculus. Among those problems were that of the center of gravity of a solid hemisphere, that of the center of gravity of a frustum of a circular paraboloid, and that of the area of a region bounded by a parabola and one of its secant lines. For explicit details of the method used, see Archimedes' use of infinitesimals.

The Sand Reckoner

- Archimedes of Syracuse

Online version: Online version [53]

Description: The first known (European) system of number-naming that can be expanded beyond the needs of everyday life.

Textbooks

Synopsis of Pure Mathematics

- G. S. Carr

Description: Contains over 6000 theorems of mathematics, assembled by George Shoobridge Carr for the purpose of training students in the art of mathematics, studied extensively by Ramanujan. (first half here) [54] It was one of the few books that attempts to summarize the entirety of known mathematics.

Arithmetick: or, The Grounde of Arts

- Robert Recorde

Description: Written in 1542, it was the first really popular arithmetic book written in the English Language.

Cocker's Arithmetick

- Edward Cocker (authorship disputed)

Description: Textbook of arithmetic published in 1678 by John Hawkins, who claimed to have edited manuscripts left by Edward Cocker, who had died in 1676. This influential mathematics textbook used to teach arithmetic in schools in the United Kingdom for over 150 years.

The Schoolmaster's Assistant, Being a Compendium of Arithmetic both Practical and Theoretical

- Thomas Dilworth

Description: An early and popular English arithmetic textbook published in America in the eighteenth century. The book reached from the introductory topics to the advanced in five sections.

Course of Pure Mathematics

- G. H. Hardy

Description: A classic textbook in introductory mathematical analysis, written by G. H. Hardy. It was first published in 1908, and went through many editions. It was intended to help reform mathematics teaching in the UK, and more specifically in the University of Cambridge, and in schools preparing pupils to study mathematics at Cambridge. As such, it was aimed directly at "scholarship level" students — the top 10% to 20% by ability. The book contains a large number of difficult problems. The content covers introductory calculus and the theory of infinite series.

Moderne Algebra

- B. L. van der Waerden

Description: The first introductory textbook (graduate level) expounding the abstract approach to algebra developed by Emil Artin and Emmy Noether. First published in German in 1931 by Springer Verlag. A later English translation was published in 1949 by Frederick Ungar Publishing Company.

Algebra

- → Saunders Mac Lane and Garrett Birkhoff

Description: A definitive introductory text for abstract algebra using a category theoretic approach. Both a rigorous introduction from first principles, and a reasonably comprehensive survey of the field.

Algebraic Geometry

- Robin Hartshorne

Description: The first comprehensive introductory (graduate level) text in algebraic geometry that used the language of schemes and cohomology. Published in 1977, it lacks aspects of the scheme language which are nowadays considered central, like the functor of points.

Naive Set Theory

- Paul Halmos

Description: An undergraduate introduction to not-very-naive set theory which has lasted for decades. It is still considered by many to be the best introduction to set theory for beginners. While the title states that it is naive, which is usually taken to mean without axioms, the book does introduce all the axioms of Zermelo-Fraenkel set theory and gives correct and rigorous definitions for basic objects. Where it differs from a "true" axiomatic set theory book is its character: There are no long-winded discussions of axiomatic minutiae, and there is next to nothing about advanced topics like large cardinals. Instead it tried, and succeeds, in being intelligible to someone who has never thought about set theory before.

Cardinal and Ordinal Numbers

- Waclaw Sierpinski

Description: The *nec plus ultra* reference for basic facts about cardinal and ordinal numbers. If you have a question about the cardinality of sets occurring in everyday mathematics, the first place to look is this book, first published in the early 1950s but based on the author's lectures on the subject over the preceding 40 years.

Set Theory: An Introduction to Independence Proofs

- Kenneth Kunen

Description: This book is not really for beginners, but graduate students with some minimal experience in set theory and formal logic will find it a valuable self-teaching tool, particularly in regard to forcing. It is far easier to read than a true reference work such as Jech, *Set Theory*. It may be the best textbook from which to learn forcing, though it has the disadvantage that the exposition of forcing relies somewhat on the earlier presentation of Martin's axiom.

Topologie

- Pavel Sergeevich Alexandrov
- → Heinz Hopf

Description: First published round 1935, this text was a pioneering "reference" text book in topology, already incorporating many modern concepts from set-theoretic topology, homological algebra and homotopy theory.

General Topology

- John L. Kelley

Description: First published in the mid-1950s,for many years the only introductory graduate level textbook in the U.S.A. teaching the basics of point set, as opposed to algebraic, topology. Prior to this the material, essential for advanced study in many fields, was only available in bits and pieces from texts on other topics or journal articles.

Topology from the Differentiable Viewpoint

- John Milnor

Description: This short book introduces the main concepts of differential topology in Milnor's lucid and concise style. While the book does not cover very much, its topics are explained beautifully in a way that illuminates all their details.

Number Theory, An approach through history from Hammurapi to Legendre

- André Weil

Description: An historical study of number theory, written by one of the 20th century's greatest researchers in the field. The book covers some thirty six centuries of arithmetical work but the bulk of it is devoted to a detailed study and exposition of the work of Fermat, Euler, Lagrange, and Legendre. The author wishes to take the reader into the workshop of his subjects to share their successes and failures. A rare opportunity to see the historical development of a subject through the mind of one of its greatest practitioners.

An Introduction to the Theory of Numbers

- G. H. Hardy and E. M. Wright

Description: This book was first published in 1938, and is still in print, with the latest edition being the 6th (2008). It is likely that almost every serious student and researcher into number theory has consulted this book, and probably has it on their bookshelf. It was not intended to be a textbook, and is rather an introduction to a wide range of differing areas of number theory which would now almost certainly be covered in separate volumes. The writing style has long been regarded as exemplary, and the approach gives insight into a variety of areas without requiring much more than a good grounding in algebra, calculus and complex numbers.

Popular writing

Gödel, Escher, Bach

- Douglas Hofstadter

Description: Gödel, Escher, Bach: an Eternal Golden Braid is a Pulitzer Prize-winning book, first published in 1979 by Basic Books. It is a book about how the creative achievements of logician Kurt Gödel, artist M. C. Escher and composer Johann Sebastian Bach interweave. As the author states: "I realized that to me, Gödel and Escher and Bach were only shadows cast in different directions by some central solid essence. I tried to reconstruct the central object, and came up with this book."

The World of Mathematics

- James R. Newman

Description: The World of Mathematics was specially designed to make mathematics more accessible to the inexperienced. It comprises nontechnical essays on every aspect of the vast subject, including articles by and about scores of eminent mathematicians, as well as literary figures, economists, biologists, and many other eminent thinkers. Includes the work of Archimedes, Galileo, Descartes, Newton, Gregor Mendel, Edmund Halley, Jonathan Swift, John Maynard Keynes, Henri Poincaré, Lewis Carroll, George Boole, Bertrand Russell, Alfred North Whitehead, John von Neumann, and many others. In addition, an informative commentary by distinguished scholar James R. Newman precedes each essay or group of essays, explaining their relevance and context in the history and development of mathematics. Originally published in 1956, it does not include many of the exciting discoveries of the later years of the 20th century but it has no equal as a general historical survey of important topics and applications.

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See also:

Higher dimensional algebra

This article is about ***higher-dimensional algebra and supercategories*** *in generalized category theory, super-category theory, and also its extensions in metamathematics*[1] . Supercategories were first introduced in 1970,[2] and were subsequently developed for applications in Theoretical Physics (especially Quantum Field Theory and Topological quantum field theory) and Mathematical Biology or Mathematical Biophysics.[3] In higher-dimensional algebra, a double → groupoid is a generalisation of a one-dimensional groupoid to two dimensions[4] , and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or n-dimensional manifolds)[5] . In general, an n-dimensional manifold is a space that locally looks like an n-dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category, followed by the more `geometric' concept of double category [6][7] [8] .

A higher level concept is that of a category of categories, or **super-category** which generalises to higher dimensions the notion of category – regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC)[9] [10] [11] [12] . Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,[13] multicategory, and multi-graph, k-partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by → Ronald Brown in 1976, in ref.[14] and were further developed towards applications in nonabelian → algebraic topology[15] [16] [17] [18] .

See also

- → Higher category theory
- Category theory
- → Algebraic topology
- Seifert–van Kampen theorem
- Abstract algebra
- Categorical algebra
- Esquisse d'un Programme
- Grothendieck's Galois theory
- Metatheory
- Metalogic
- Metamathematics
- Colored graphs
- Multicategory
- Enriched category

Further reading

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Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets, and (n+1)-categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, → homotopy theory), where "weak" structures arise in the form of higher categories.[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are no longer strict (that is, they are not given by equalities), but rather are satisfied up to an isomorphism of the next level. An example in → topology is the composition of paths, which is associative only up to homotopy. These isomorphisms must well behave between them and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories, were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, so that bicategories can be said to be "monoidal categories with many objects." Weak 3-categories, also called tricategories, and higher-level generalizations are increasingly harder to define explicitly. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

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[1] Leinster, pp 18-19
[2] Baez, p 6

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See also

- Network Science
- Polytely
- Higher-dimensional algebra

External links

- John Baez Tale of *n*-Categories (http://math.ucr.edu/home/baez/week73.html)

Van Kampen's theorem

In mathematics, the **Seifert–van Kampen theorem** of → algebraic topology, sometimes just called **van Kampen's theorem**, expresses the structure of the → fundamental group of a topological space X, in terms of the fundamental groups of two open, path-connected subspaces U and V that cover X. It can therefore be used for computations of the fundamental group of spaces that are constructed out of simpler ones.

The underlying idea is that paths in X can be partitioned: into journeys through the intersection W of U and V, through U but outside V, and through V outside U. In order to move segments of paths around, by homotopy to form loops returning to a base point w in W, we should assume U, V and W are path-connected; and that W isn't empty. We assume also that U and V are open subspaces with union X.

Under these conditions, $\pi_1(U,w)$, $\pi_1(V,w)$, and $\pi_1(W,w)$, together with the inclusion homomorphisms (induced by the inclusion map):

$$I\colon \pi_1(W,w) \to \pi_1(U,w)$$

and

$$J\colon \pi_1(W,w) \to \pi_1(V,w)$$

are sufficient data to determine $\pi_1(X,w)$. The maps I and J extend to an epimorphism

$$\Phi\colon \pi_1(U,w) * \pi_1(V,w) \to \pi_1(X,w)$$

where $\pi_1(U,w) * \pi_1(V,w)$ is the free product of $\pi_1(U,w)$ and $\pi_1(V,w)$. The kernel of the map Φ are the loops in W that, when viewed in X , are homotopic to the trivial one at w. The group $\pi_1(X,w)$ is therefore isomorphic to $\pi_1(U,w) * \pi_1(V,w)$ modulo such elements, more precisely, to the almagamated free product: $\pi_1(U,w)*_{\pi(W,w)}\pi_1(V,w)$

In particular, when W is simply connected (so that its fundamental group is the trivial group), the theorem says that $\pi_1(X,w)$ is isomorphic to the free product $\pi_1(U,w) * \pi_1(V,w)$.

Equivalent formulations

In the language of combinatorial group theory, π_1(X,w) is the free product with amalgamation of those of U and V, with respect to the homomorphisms I and J (which might not be injective): given group presentations

$\pi_1(U,w) = \langle u_1,\ldots,u_k \mid \alpha_1,\ldots,\alpha_l\rangle$

$\pi_1(V,w) = \langle v_1,\ldots,v_m \mid \beta_1,\ldots,\beta_n\rangle$

$\pi_1(W,w) = \langle w_1,\ldots,w_p \mid \gamma_1,\ldots,\gamma_q\rangle$

the amalgamation can be written in terms of generators and relations as $\pi_1(X,w) = \langle u, v \mid \alpha, \beta, \gamma, I(w_r)\cdot J(w_r)^{-1}\rangle$ where each letter u, v, w, α, β, γ stands for the respective set of generators or relators, and the final relator means that the images of each generator w_r under the inclusions I, J are equivalent in the fundamental group of X.

In category theory, the fundamental group of X is a colimit of the diagram of those of U, V and W. More precisely, π_1(X,w) is the pushout of the diagram.

Van Kampen's theorem for fundamental groups

Van Kampen's theorem for fundamental groups[1] :

Let X be a topological space which is the union of the interiors of two path connected subspaces X_1, X_2. *Suppose* $X_0 := X_1 \cap X_2$ *is path connected. Let also* $* \in X_0$ *and* $i_k : \pi_1(X_0,*) \to \pi_1(X_k,*)$, $j_k : \pi_1(X_k,*) \to \pi_1(X,*)$ *be induced by the inclusions for k= 1, 2. Then X is path connected and the natural morphism* $\pi_1(X_1,*) *_{\pi_1(X_0,*)} \pi_1(X_2,*) \to \pi_1(X,*)$ *is an isomorphism*, that is, *the fundamental group of X is the free product of the fundamental groups of* X_1 *and* X_2 *with amalgamation* of $\pi_1(X_0,*)$.

Usually the morphisms induced by inclusion in this theorem are not themselves injective, and the more precise version of the statement is in terms of pushouts of groups. The notion of pushout in the category of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X,A)$ on a set A of base points,[2] . *This groupoid consists of homotopy classes relative to the end points of paths in X joining points of* $A \cap X$ *. In particular, if X is a contractible space, and A consists of two distinct points of X, then* $\pi_1(X,A)$ *is easily seen to be isomorphic to the groupoid often written* $\mathcal{I}$ *with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups*[3] .

Theorem: *Let the topological space X be covered by the interiors of two subspaces* X_1, X_2 *and let A be a set which meets each path component of* X_1, X_2 *and* $X_0 := X_1 \cap X_2$*. Then A meets each path component of X and the diagram **P** of morphisms induced by inclusion*

$$\begin{array}{ccc} \pi_1(X_0,A) & \xrightarrow{\pi_1(i_1)} & \pi_1(X_1,A) \\ {\scriptstyle \pi_1(i_2)}\downarrow & & \downarrow{\scriptstyle \pi_1(j_1)} \\ \pi_1(X_2,A) & \xrightarrow[\pi_1(j_2)]{} & \pi_1(X,A) \end{array}$$

is a pushout diagram in the category of groupoids.[4]

The interpretation of this theorem as a calculational tool for fundamental groups needs some development of `combinatorial groupoid theory',[5] [6] . This theorem implies the calculation of the fundamental group of the circle as the group of integers, since the group of integers is obtained from the groupoid $\mathcal{I}$ by identifying, in the category of groupoids, its two vertices.

There is a version of the last theorem when X is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets[7] [8] . The conclusion is that if A meets each path component of all 1,2,3-fold intersections of the sets U_λ , then A meets all path components of X and the diagram $\bigsqcup_{(\lambda,\mu)\in\Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \rightrightarrows \bigsqcup_{\lambda\in\Lambda} \pi_1(U_\lambda, A) \to \pi_1(X,A)$ of morphisms induced by inclusions is a coequaliser in the category of groupoids.

Examples

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces. For example, consider the sphere S^2. Pick open sets $A = S^2 - n$ and $B = S^2 - s$ where n and s denote the north and south poles respectively. Then we have the property that A, B and A ⋂ B are open path connected sets. Thus we can see that there is a commutative diagram including A ⋂ B into A and B and then another inclusion from A and B into S^2 and that there is a corresponding diagram of homomorphisms between the fundamental groups of each subspace. Applying Van Kampen's theorem gives the result $\pi_1(S^2) = \pi_1(A) * \pi_1(B)/ker(\Phi)$. However A and B are both homeomorphic to $\mathbf{R^2}$ which is simply connected, so both A and B have trivial fundamental groups. It is clear from this that the fundamental group of S^2 is trivial.

A more complicated example is the calculation of the fundamental group of a genus *n* orientable surface *S*, otherwise known as the *genus n surface group*. One can construct *S* using its standard fundamental polygon. For the first open set *A*, pick a disk within the center of the polygon. Pick *B* to be the complement in *S* of the center point of *A*. Then the intersection of *A* and *B* is an annulus, which is known to be homotopy equivalent to (and so has the same fundamental group as) a circle. Then $\pi_1(A \cap B) = \pi_1(S^1)$, which is the integers, and $\pi_1(A) = \pi_1(D^2) = 1$. Thus the inclusion of $\pi_1(A \cap B)$ into $\pi_1(A)$ sends any generator to the trivial element. However, the inclusion

of $\pi_1(A \cap B)$ into $\pi_1(B)$ is not trivial. In order to understand this, first one must calculate $\pi_1(B)$. This is easily done as one ca retract B (which is *S* with one point deleted) onto the edges labeled by $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}\ldots A_nB_nA_n^{-1}B_n^{-1}$. This space is known to be the wedge sum of 2*n* circles (also called a bouquet of circles), which further is known to have fundamental group isomorphic to the free group with 2*n* generators, which in this case can be represented by the edges themselves: $\{A_1, B_1, \ldots, A_n, B_n\}$. We now have enough information to apply Van Kampen's theorem. The genera loops $\{A_1, B_1, \ldots, A_n, B_n\}$ (*A* is simply connected, so it contributes no generators) and there is exactly on $A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1}\ldots A_nB_nA_n^{-1}B_n^{-1} = 1$. Using generators and relations, this group is denoted

$$\langle A_1, B_1, \ldots, A_n, B_n | A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_n B_n A_n^{-1} B_n^{-1} \rangle.$$

Generalizations

This theorem has been extended to the non-connected case by using the fundamental groupoid π_1(X,A) on a *set A of base points*, which consists of homotopy classes of paths in X joining points of X which lie in A. The connectivity conditions for the theorem then become that A meets each path-component of U,V,W. The pushout is now in the category of groupoids. This extended theorem allows the determination of the fundamental group of the circle, and many other useful cases. For example, if the intersection W has two path components, it is convenient to let A consist of one point in each of these components. A theorem for arbitrary covers, with the restriction that A meets all three fold intersections of the sets of the cover, is given in the paper by Brown and Razak cited below. Applications of the fundamental groupoid on a set of base points to the Jordan curve theorem, Covering space, and orbit space are given in Ronald Brown's book cited below.

In the case of orbit spaces, it is convenient to take A to include all the fixed points of the action. An example here is the conjugation action on the circle.

The version that allows more than two overlapping sets but with A a singleton is also given in Allen Hatcher's book below, theorem 1.20.

In fact, we can extend van Kampen's theorem significantly further by considering the → fundamental groupoid $\Pi(X)$, an element of the category of small categories whose objects are points of X and whose arrows are paths between points modulo homotopy equivalence. In this case, to determine the fundamental groupoid of a space, we need only know the fundamental groupoids of the open sets covering X as follows: create a new category in which the objects are fundamental groupoids of the open sets, with an arrow between groupoids if the domain space is a subspace of the codomain. Then van Kampen's theorem is the assertion that the fundamental groupoid of X is the colimit of the diagram. For details, see Peter May's book, chapter 2.

References to higher dimensional versions of the theorem which yield some information on homotopy types are given in an article on higher dimensional group theories and groupoids.[9]

See also

- → Higher dimensional algebra
- → Higher category theory
- → Alexander Grothendieck
- Van Kampen
- → Ronald Brown (mathematician)

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Groupoid

In abstract algebra, a branch of mathematics, especially in category theory and → homotopy theory, a **groupoid** (less often **Brandt groupoid** or **virtual group**) generalises the notion of group and of category in several equivalent ways. A groupoid can be seen as a:

- *Group* with a partial function replacing the binary operation;
- *Category* in which every morphism is an isomorphism. A category of this sort can be viewed as augmented with a unary operation, called *inverse* by analogy with group theory.

Special cases include:

- *Setoids*, that is: sets which come with an equivalence relation;
- *G-sets*, sets equipped with an action of a group *G*.

Groupoids are often used to reason about geometrical objects such as manifolds. Heinrich Brandt introduced groupoids implicitly via Brandt semigroups in 1926.[1]

Definitions

Algebraic

A groupoid is a set *G* with a unary operation $^{-1}: G \to G,$ and a partial function $*: G \times G \to G.$ * is not a binary operation because it is not necessarily defined for all possible pairs of *G*-elements. The precise conditions under which * is defined are not articulated here and vary by situation.
$*$ and $^{-1}$ have the following axiomatic properties. Let *a*, *b*, and *c* be elements of *G*. Then:

- *Associativity*: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal. Conversely, if either of these last two expressions is defined, then so is the other (and again they are equal).
- *Inverse*: $a^{-1} * a$ and $a * a^{-1}$ are always defined.
- *Identity*: If $a * b$ is defined, then $a * b * b^{-1} = a$, and $a^{-1} * a * b = b$. (The previous two axioms already show that these expressions are defined and unambiguous.)

In short:

- $(a * b) * c = a * (b * c)$;
- $(a * b) * b^{-1} = a$;
- $a^{-1} * (a * b) = b$.

From these axioms, two easy and convenient theorems follow:

- $(a^{-1})^{-1} = a$;
- If $a * b$ is defined, then $(a * b)^{-1} = b^{-1} * a^{-1}$.

Category theoretic

A groupoid is a small category in which every morphism is an isomorphism, and hence invertible. More precisely, a groupoid *G* is:

- A set G_0 of *objects*;
- For each pair of objects *x* and *y* in G_0, there exists a (possibly empty) set $G(x,y)$ of *morphisms* (or *arrows*) from *x* to *y*. We write $f : x \to y$ to indicate that *f* is an element of $G(x,y)$.

The objects and morphisms have the properties:

- For every object *x*, there exists the element id_x of $G(x,x)$;

- For each triple of objects x, y, and z, there exists the function $\mathrm{comp}_{x,y,z} : G(x,y) \times G(y,z) \to G(x,z)$. We write gf for $\mathrm{comp}_{x,y,z}(f,g)$, where $f \in G(x,y)$, and $g \in G(y,z)$;
- There exists the function $\mathrm{inv}_{x,y} : G(x,y) \to G(y,x)$.

Moreover, if $f : x \to y$, $g : y \to z$, and $h : z \to w$, then:

- $f\mathrm{id}_x = f$ and $\mathrm{id}_y f = f$;
- $(hg)f = h(gf)$;
- $f\mathrm{inv}(f) = \mathrm{id}_y$ and $\mathrm{inv}(f)f = \mathrm{id}_x$.

If f is an element of $G(x,y)$ then x is called the **source** of f, written $s(f)$, and y the **target** of f (written $t(f)$).

Comparing the definitions

The algebraic and category-theoretic definitions are equivalent, as follows. Given a groupoid in the category-theoretic sense, let G be the disjoint union of all of the sets $G(x,y)$ (i.e. the sets of morphisms from x to y). Then comp and inv become partially defined operations on G, and inv will in fact be defined everywhere; so we define * to be comp and $^{-1}$ to be inv. Thus we have a groupoid in the algebraic sense. Explicit reference to G_0 (and hence to id) can be dropped.

Conversely, given a groupoid G in the algebraic sense, with typical element f, let G_0 be the set of all elements of the form $f*f^{-1}$. In other words, the objects are identified with the identity morphisms, so that id_x is just x. Let $G(x,y)$ be the set of all elements f such that $y\,f\,x$ is defined. Then $^{-1}$ and * break up into several functions on the various $G(x,y)$, which may be called inv and comp, respectively.

Sets in the definitions above may be replaced with classes, as is generally the case in category theory.

Vertex groups

Given a groupoid G, the **vertex groups** or **isotropy group** in G are the subsets of the form $G(x,x)$, where x is any object of G. It follows easily from the axioms above that these are indeed groups, as every pair of elements is composable and inverses are in the same vertex group.

Groupoid Category

The category whose objects are groupoids and whose morphisms are groupoid homomorphisms is called the **groupoid category**, or the **category of groupoids**.

Examples

Linear algebra

Given a field K, the corresponding **general linear groupoid** $GL_*(K)$ consists of all invertible matrices whose entries range over K. Matrix multiplication interprets composition. If $G = GL_*(K)$, then the set of natural numbers is a proper subset of G_0, since for each natural number n, there is a corresponding identity matrix of dimension n. $G(m,n)$ is empty unless $m=n$, in which case it is the set of all nxn invertible matrices.

Topology

Given a topological space X, let G_0 be the set X. The morphisms from the point p to the point q are equivalence classes of continuous paths from p to q, with two paths being equivalent if they are homotopic. Two such morphisms are composed by first following the first path, then the second; the homotopy equivalence guarantees that this composition is associative. This groupoid is called the → fundamental groupoid of X, denoted $\pi_1(X)$. The usual fundamental group $\pi_1(X,x)$ is then the vertex group for the point x.

An important extension of this idea is to consider the fundamental groupoid $\pi_1(X,A)$ where A is a set of "base points" and a subset of X. Here, one considers only paths whose endpoints belong to A. $\pi_1(X,A)$ is a sub-groupoid of $\pi_1(X)$. The set A may be chosen according to the geometry of the situation at hand.

Equivalence relation

If X is a set with an equivalence relation denoted by infix $\sim$, then a groupoid "representing" this equivalence relation can be formed as follows:

- The objects of the groupoid are the elements of X;
- For any two elements x and y in X, there is a single morphism from x to y if and only if $x \sim y$.

Group action

If the group G acts on the set X, then we can form the **action groupoid** representing this group action as follows:

- The objects are the elements of X;
- For any two elements x and y in X, there is a morphism from x to y corresponding to every element g of G such that $gx = y$;
- Composition of morphisms interprets the binary operation of G.

More explicitly, the *action groupoid* is the set $G \times X$ with source and target maps $s(g,x) = x$ and $t(g,x) = gx$. It is often denoted $G \ltimes X$ (or $X \rtimes G$). Multiplication (or composition) in the groupoid is then $(h,y)(g,x) = (hg,x)$ which is defined provided $y=gx$.

For x in X, the vertex group consists of those (g,x) with $gx = x$, which is just the isotropy subgroup at x for the given action (which is why vertex groups are also called isotropy groups).

Another way to describe G-sets is the functor category $[\mathrm{Gr}, \mathrm{Set}]$, where Gr is the groupoid (category) with one element and isomorphic to the group G. Indeed, every functor F of this category defines a set $X=F(\mathrm{Gr})$ and for every g in G (i.e. for every morphism in Gr) induces a bijection $F_g : X \to X$. The categorical structure of the functor F assures us that F defines a G-action on the set X. The (unique) representable functor $F : \mathrm{Gr} \to \mathrm{Set}$ is the Cayley representation of G. In fact, this functor is isomorphic to $\mathrm{Hom}(\mathrm{Gr}, -)$ and so sends $\mathrm{ob}(\mathrm{Gr})$ to the set $\mathrm{Hom}(\mathrm{Gr}, \mathrm{Gr})$ which is by definition the "set" G and the morphism g of Gr (i.e. the element g of G) to the permutation F_g of the set G. We deduce from the Yoneda embedding that the group G is isomorphic to the group $\{F_g \mid g \in G\}$, a subgroup of the group of permutations of G.

Fifteen puzzle

The symmetries of the Fifteen puzzle form a groupoid (not a group, as not all moves can be composed). This groupoid acts on configurations.

Relation to groups

Group-like structures				
	Totality	**Associativity**	**Identity**	**Inverses**
Group	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
→ Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

If a groupoid has only one object, then the set of its morphisms forms a group. Using the algebraic definition, such a groupoid is literally just a group. Many concepts of group theory generalize to groupoids, with the notion of functor replacing that of group homomorphism.

If x is an object of the groupoid G, then the set of all morphisms from x to x forms a group $G(x)$. If there is a morphism f from x to y, then the groups $G(x)$ and $G(y)$ are isomorphic, with an isomorphism given by the mapping $g \to fgf^{-1}$.

Every connected groupoid (that is, one in which any two objects are connected by at least one morphism) is isomorphic to a groupoid of the following form. Pick a group G and a set (or class) X. Let the objects of the groupoid be the elements of X. For elements x and y of X, let the set of morphisms from x to y be G. Composition of morphisms is the group operation of G. If the groupoid is not connected, then it is isomorphic to a disjoint union of groupoids of the above type (possibly with different groups G for each connected component). Thus any groupoid may be given (up to isomorphism) by a set of ordered pairs (X,G).

Note that the isomorphism described above is not unique, and there is no natural choice. Choosing such an isomorphism for a connected groupoid essentially amounts to picking one object x_0, a group isomorphism h from $G(x_0)$ to G, and for each x other than x_0, a morphism in G from x_0 to x.

In category-theoretic terms, each connected component of a groupoid is equivalent (but not isomorphic) to a groupoid with a single object, that is, a single group. Thus any groupoid is equivalent to a multiset of unrelated groups. In other words, for equivalence instead of isomorphism, one need not specify the sets X, only the groups G.

Consider the examples in the previous section. The general linear groupoid is both equivalent and isomorphic to the disjoint union of the various general linear groups $GL_n(F)$. On the other hand:

- The fundamental groupoid of X is equivalent to the collection of the → fundamental groups of each path-connected component of X, but an isomorphism requires specifying the set of points in each component;
- The set X with the equivalence relation $\sim$ is equivalent (as a groupoid) to one copy of the trivial group for each equivalence class, but an isomorphism requires specifying what each equivalence class is:
- The set X equipped with an action of the group G is equivalent (as a groupoid) to one copy of G for each orbit of the action, but an isomorphism requires specifying what set each orbit is.

The collapse of a groupoid into a mere collection of groups loses some information, even from a category-theoretic point of view, because it is not natural. Thus when groupoids arise in terms of other structures, as in the above examples, it can be helpful to maintain the full groupoid. Otherwise, one must choose a way to view each $G(x)$ in terms of a single group, and this choice can be arbitrary. In our example from → topology, you would have to make a coherent choice of paths (or equivalence classes of paths) from each point p to each point q in the same path-connected component.

As a more illuminating example, the classification of groupoids with one endomorphism does not reduce to purely group theoretic considerations. This is analogous to the fact that the classification of vector spaces with one endomorphism is nontrivial.

Morphisms of groupoids come in more kinds than those of groups: we have, for example, fibrations, covering morphisms, universal morphisms, and quotient morphisms. Thus a subgroup H of a group G yields an action of G on the set of cosets of H in G and hence a covering morphism p from, say, K to G, where K is a groupoid with vertex groups isomorphic to H. In this way, presentations of the group G can be "lifted" to presentations of the groupoid K, and this is a useful way of obtaining information about presentations of the subgroup H. For further information, see the books by Higgins and by Brown in the References.

Another useful fact is that the category of groupoids, unlike that of groups, is cartesian closed.

Lie groupoids and Lie algebroids

When studying geometrical objects, the arising groupoids often carry some differentiable structure, turning them into → Lie groupoids. These can be studied in terms of → Lie algebroids, in analogy to the relation between Lie groups and Lie algebras.

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Lie groupoid

In mathematics, a **Lie groupoid** is a → groupoid where the set Ob of objects and the set Mor of morphisms are both manifolds, the source and target operations

$$s, t : Mor \to Ob$$

are submersions, and all the category operations (source and target, composition, and identity-assigning map) are smooth.

A Lie groupoid can thus be thought of as a "many-object generalization" of a Lie group, just as a groupoid is a many-object generalization of a group. Just as every Lie group has a Lie algebra, every Lie groupoid has a → Lie algebroid.

Examples

- Any Lie group gives a Lie groupoid with one object, and conversely. So, the theory of Lie groupoids includes the theory of Lie groups.
- Given any manifold M, there is a Lie groupoid called the pair groupoid, with M as the manifold of objects, and precisely one morphism from any object to any other. In this Lie groupoid the manifold of morphisms is thus $M \times M$.
- Given a Lie group G acting on a manifold M, there is a Lie groupoid called the translation groupoid with one morphism for each triple $g \in G, x, y \in M$ with $gx = y$.
- Any foliation gives a Lie groupoid.
- Any principal bundle $P \to M$ with structure group *G* gives a groupoid, namely $P \times P/G$ over *M*, where *G* acts on the pairs componentwise. Composition is defined via compatible representatives as in the pair groupoid.

Morita Morphisms and Smooth Stacks

Beside isomorphism of groupoids there is a more coarse notation of equivalence, the so called Morita equivalence. A quite general example is the Morita-morphism of the **Čech groupoid** which goes as follows. Let *M* be a smooth manifold and $\{U_\alpha\}$ an open cover of *M*. Define $G_0 := \coprod_\alpha U_\alpha$ the disjoint union with the obvious submersion $p : G_0 \to M$. In order to encode the structure of the manifold *M* define the set of morphisms $G_1 := \coprod_{\alpha,\beta} U_{\alpha\beta}$ where $U_{\alpha\beta} = U_\alpha \cap U_\beta \subset M$. The source and target map are defined as the embeddings $s : U_{\alpha\beta} \to U_\alpha$ and $t : U_{\alpha\beta} \to U_\beta$. And multiplication is the obvious one if we read the $U_{\alpha\beta}$ as subsets of *M* (compatible points in $U_{\alpha\beta}$ and $U_{\beta\gamma}$ actually are the same in *M* and also lie in $U_{\alpha\gamma}$).

This Čech groupoid is in fact the pullback groupoid of $M \Rightarrow M$, i.e. the trivial groupoid over *M*, under *p*. That is what makes it Morita-morphism.

In order to get the notion of an equivalence relation we need to make the construction symmetric and show that it is also transitive. In this sense we say that 2 groupoids $G_1 \Rightarrow G_0$ and $H_1 \Rightarrow H_0$ are Morita equivalent iff there

exists a third groupoid $K_1 \Rightarrow K_0$ together with 2 Morita morphisms from *G* to *K* and *H* to *K*. Transitivity is an interesting construction in the category of groupoid principal bundles and left to the reader.

It arises the question of what is preserved under the Morita equivalence. There are 2 obvious things, one the coarse quotient/ orbit space of the groupoid $G_0/G_1 = H_0/H_1$ and secondly the stabilizer groups $G_p \cong H_q$ for corresponding points $p \in G_0$ and $q \in H_0$.

The further question of what is the structure of the coarse quotient space leads to the notion of a smooth stack. We can expect the coarse quotient to be a smooth manifold if for example the stabilizer groups are trivial (as in the example of the Čech groupoid). But if the stabilizer groups change we cannot expect a smooth manifold any longer. The solution is to revert the problem and to define:

A **smooth stack** is a Morita-equivalence class of Lie groupoids. The natural geometric objects living on the stack are the geometric objects on Lie groupoids invariant under Morita-equivalence. As an example consider the Lie groupoid cohomology.

Examples

- The notion of smooth stack is quite general, obviously all smooth manifolds are smooth stacks.
- But also orbifolds are smooth stacks, namely (equivalence classes of) étale groupoids.
- Orbit spaces of foliations are another class of examples

External links

Alan Weinstein, Groupoids: unifying internal and external symmetry, *AMS Notices*, **43** (1996), 744-752. Also available as arXiv:math/9602220 [1]

Kirill Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge U. Press, 1987.

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[1] http://arxiv.org/abs/math/9602220

Lie algebroid

In mathematics, Lie algebroids serve the same role in the theory of → Lie groupoids that Lie algebras serve in the theory of Lie groups: reducing global problems to infinitesimal ones. Just as a Lie groupoid can be thought of as a "Lie group with many objects", a Lie algebroid is like a "Lie algebra with many objects".

More precisely, a **Lie algebroid** is a triple $(E, [\cdot,\cdot], \rho)$ consisting of a vector bundle E over a manifold M, together with a Lie bracket $[\cdot,\cdot]$ on its module of sections $\Gamma(E)$ and a morphism of vector bundles $\rho : E \to TM$ called the **anchor**. Here TM is the tangent bundle of M. The anchor and the bracket are to satisfy the Leibniz rule:

$$[X, fY] = \rho(X)f \cdot Y + f[X, Y]$$

where $X, Y \in \Gamma(E), f \in C^\infty(M)$ and $\rho(X)f$ is the derivative of f along the vector field $\rho(X)$. It follows that

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

for all $X, Y \in \Gamma(E)$.

Examples

- Every Lie algebra is a Lie algebroid over the one point manifold.
- The tangent bundle TM of a manifold M is a Lie algebroid for the Lie bracket of vector fields and the identity of TM as an anchor.
- Every integrable subbundle of the tangent bundle — that is, one whose sections are closed under the Lie bracket — also defines a Lie algebroid.
- Every bundle of Lie algebras over a smooth manifold defines a Lie algebroid where the Lie bracket is defined pointwise and the anchor map is equal to zero.
- To every → Lie groupoid is associated a Lie algebroid, generalizing how a Lie algebra is associated to a Lie group (see also below). For example, the Lie algebroid TM comes from the pair groupoid whose objects are M , with one isomorphism between each pair of objects. Unfortunately, going back from a Lie algebroid to a Lie groupoid is not always possible[1] , but every Lie algebroid gives a stacky Lie groupoid[2] [3] .
- Given the action of a Lie algebra g on a manifold M, the set of g -invariant vector fields on M is a Lie algebroid over the space of orbits of the action.
- **Atiyah algebroid**. Given a vector bundle V over a smooth manifold M consider its derivations, i.e. smooth $\mathbb{R}$ -linear maps $\psi : \Gamma(V) \to \Gamma(V)$ for which exists a vector field *X* such that they fulfill the Leibniz rule $\psi(fv) = X[f]v + f\psi(v)$ for all smooth functions *f* and all sections *v* into the vector bundle. The association $\psi \to X$ is clearly linear and thus comes from a map of vector bundles $\rho : A(V) \to TM$ (if you find the bundle whose sections give the derivations). The Atiyah algebroid is further characterized by fitting into the following short exact sequence: $0 \to \mathrm{End}_M(V) \to A(V) \to TM \to 0$ To see that the Atiyah algebroid exists for every vector bundle note that it is the Lie algebroid associated to the Lie groupoid coming from the frame bundle of the vector bundle *V*.

Lie algebroid associated to a Lie groupoid

To describe the construction let us fix some notation. *G* is the space of morphisms of the Lie groupoid, *M* the space of objects, $e: M \to G$ the units and $t: G \to M$ the target map.

$T^tG = \bigcup_{p \in M} T(t^{-1}(p)) \subset TG$ the *t*-fiber tangent space. The Lie algebroid is now the vector bundle $A := e^*T^tG$. This inherits a bracket from *G*, because we can identify the *M*-sections into *A* with left-invariant vector fields on *G*. Further these sections act on the smooth functions of *M* by identifying these with left-invariant functions on *G*.

As a more explicit example consider the Lie algebroid associated to the pair groupoid $G := M \times M$. The target map is $t: G \to M : (p,q) \mapsto p$ and the units $e: M \to G : p \mapsto (p,p)$. The *t*-fibers are $p \times M$ and therefore $T^tG = \bigcup_{p \in M} p \times TM \subset TM \times TM$. So the Lie algebroid is the vector bundle $A := e^*T^tG = \bigcup_{p \in M} T_pM = TM$. The extension of sections *X* into *A* to left-invariant vector fields on *G* is simply $\tilde{X}(p,q) = 0 \oplus X(q)$ and the extension of a smooth function *f* from *M* to a left-invariant function on *G* is $\tilde{f}(p,q) = f(q)$. Therefore the bracket on *A* is just the Lie bracket of tangent vector fields and the anchor map is just the identity.

Of course you could do an analog construction with the source map and right-invariant vector fields/ functions. However you get an isomorphic Lie algebroid, with the explicit isomorphism i_*, where $i: G \to G$ is the inverse map.

External links

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symmetry, *AMS Notices*, **43** (1996), 744-752. Also available as arXiv:math/9602220 [1]

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Grothendieck topology

In category theory, a branch of mathematics, a **Grothendieck topology** is a structure on a category C which makes the objects of C act like the open sets of a topological space. A category together with a choice of Grothendieck topology is called a **site**.

Grothendieck topologies axiomatize the notion of an open cover. Using the notion of covering provided by a Grothendieck topology, it becomes possible to define sheaves on a category and their cohomology. This was first done in algebraic geometry and algebraic number theory by → Alexander Grothendieck to define the étale cohomology of a scheme. It has been used to define other cohomology theories since then, such as l-adic cohomology, flat cohomology, and crystalline cohomology. While Grothendieck topologies are most often used to define cohomology theories, they have found other applications as well, such as to John Tate's theory of rigid analytic geometry.

There is a natural way to associate a site to an ordinary topological space, and Grothendieck's theory is loosely regarded as a generalization of classical topology. Under meager point-set hypotheses, namely sobriety, this is completely accurate—it is possible to recover a sober space from its associated site. However simple examples such as the indiscrete topological space show that not all topological spaces can be expressed using Grothendieck topologies. Conversely, there are Grothendieck topologies which do not come from topological spaces.

Introduction

André Weil's famous Weil conjectures proposed that certain properties of equations with integral coefficients should be understood as geometric properties of the algebraic variety that they defined. His conjectures postulated that there should be a cohomology theory of algebraic varieties which gave number-theoretic information about their defining equations. This cohomology theory was known as the "Weil cohomology", but using the tools he had available, Weil was unable to construct it.

In the early 1960s, Alexander Grothendieck introduced étale maps into algebraic geometry as algebraic analogues of local analytic isomorphisms in analytic geometry. He used étale coverings to define an algebraic analogue of the → fundamental group of a topological space. Soon → Jean-Pierre Serre noticed that some properties of étale coverings mimicked those of open immersions, and that consequently it was possible to make constructions which imitated the cohomology functor H^1. Grothendieck saw that it would be possible to use Serre's idea to define a cohomology theory which he suspected would be the Weil cohomology. To define this cohomology theory, Grothendieck needed to replace the usual, topological notion of an open covering with one that would use étale coverings instead. Grothendieck also saw how to phrase the definition of covering abstractly; this is where the definition of a Grothendieck topology comes from.

Definition

Motivation

The classical definition of a sheaf begins with a topological space X. A sheaf associates information to the open sets of X. This information can be phrased abstractly by letting $O(X)$ be the category whose objects are the open subsets U of X and whose morphisms are the inclusion maps $U \to V$ of open sets U and V of X. We will call such maps *open immersions*, just as in the context of schemes. Then a presheaf on X is a contravariant functor from $O(X)$ to the category of sets, and a sheaf is a presheaf which satisfies the gluing axiom. The gluing axiom is phrased in terms of pointwise covering, i.e., $\{U_i\}$ covers U if and only if $\bigcup_i U_i = U$. Because the value of a presheaf on an open set determines the value on any smaller set by restriction, it is equivalent to work with the collection of all open subsets V contained in some U_i. A Grothendieck topology encodes the information about collections of open subsets of U

and which collections cover U without any reference to the space itself.

Sieves

In a Grothendieck topology, the notion of a collection of open subsets of U stable under inclusion is replaced by the notion of a sieve. If c is any given object in C, a **sieve** on c is a subfunctor of the functor Hom(−, c); (this is the Yoneda embedding applied to c). In the case of $O(X)$, a sieve S on an open set U corresponds to a collection of open subsets of U "selected" by S. More precisely, consider that for any open subset V of U, $S(V)$ will be a subset of Hom(V, U), which has only one element, the open immersion $V \to U$. Then V will be considered "selected" by S if and only if $S(V)$ is nonempty.

If S is a sieve on X, and f: $Y \to X$ is a morphism, then left composition by f gives a sieve on Y called the **pullback of** S **along** f, denoted by $f^* S$. It is defined as the fibered product $S \times_{\mathrm{Hom}(-,\, X)} \mathrm{Hom}(-, Y)$ together with its natural embedding in Hom(−, Y). More concretely, for each object Z of C, $f^* S(Z) = \{ g\colon Z \to Y \mid fg \in S(Z) \}$, and $f^* S$ inherits its action on morphisms by being a subfunctor of Hom(−, Y). In the classical example, the pullback of a collection $\{V_i\}$ of subsets of U along an inclusion $W \to U$ is the collection $\{V_i \cap W\}$.

Covering Sieves

A classical topology on a set X is a collection of distinguished subsets, called open sets. This selection is subject to certain conditions: the axioms of a topological space. By comparison, a Grothendieck topology J on a category C is a collection, *for each object c of* C, of distinguished sieves on c, called **covering sieves** of c and denoted by $J(c)$. This selection will be subject to certain axioms, stated below. Continuing the previous example, a sieve S on an open set U in $O(X)$ will be a covering sieve if and only if the union of all the open sets V for which $S(V)$ is nonempty equals U; in other words, if and only if S gives us a collection of open sets which cover U in the classical sense.

Axioms

The conditions we impose on a **Grothendieck topology** are:

- (T 1) (Base change) If S is a covering sieve on X, and f: $Y \to X$ is a morphism, then the pullback $f^* S$ is a covering sieve on Y.
- (T 2) (Local character) Let S be a covering sieve on X, and let T be any sieve on X. Suppose that for each object Y of C and each arrow f: $Y \to X$ in $S(Y)$, the pullback sieve $f^* T$ is a covering sieve on Y. Then T is a covering sieve on X.
- (T 3) (Identity) Hom(−, X) is a covering sieve on X for any object X in C.

The base change axiom corresponds to the idea that if $\{ U_i \}$ covers U, then $\{U_i \cap V\}$ should cover $U \cap V$. The local character axiom corresponds to the idea that if $\{U_i\}$ covers U and $\{V_{ij}\}_{j \in J_i}$ covers U_i for each i, then the collection $\{V_{ij}\}$ for all i and j should cover U. Lastly, the identity axiom corresponds to the idea that any set is covered by all its possible subsets.

Alternative Axioms

In fact, it is possible to put these axioms in another form where their geometric character is more apparent, assuming that the underlying category C contains certain fibered products. In this case, instead of specifying sieves, we can specify that certain collections of maps with a common codomain should cover their codomain. These collections are called **covering families**. If the collection of all covering families satisfies certain axioms, then we say that they form a **Grothendieck pretopology**. These axioms are:

- (PT 0) (Existence of fibered products) For all objects X of C, and for all morphisms $X_0 \to X$ which appear in some covering family of X, and for all morphisms $Y \to X$, the fibered product $X_0 \times_X Y$ exists.

- (PT 1) (Stability under base change) For all objects X of C, all morphisms $Y \to X$, and all covering families $\{X_\alpha \to X\}$, the family $\{X_\alpha \times_X Y \to Y\}$ is a covering family.
- (PT 2) (Local character) If $\{X_\alpha \to X\}$ is a covering family, and if for all α, $\{X_{\beta\alpha} \to X_\alpha\}$ is a covering family, then the family of composites $\{X_{\beta\alpha} \to X_\alpha \to X\}$ is a covering family.
- (PT 3) (Isomorphisms) If f: $Y \to X$ is an isomorphism, then $\{f\}$ is a covering family.

For any pretopology, the collection of all sieves that contain a covering family from the pretopology is always a Grothendieck topology.

For categories with fibered products, there is a converse. Given a collection of arrows $\{X_\alpha \to X\}$, we construct a sieve S by letting $S(Y)$ be the set of all morphisms $Y \to X$ that factor through some arrow $X_\alpha \to X$. This is called the sieve **generated by** $\{X_\alpha \to X\}$. Now choose a topology. Say that $\{X_\alpha \to X\}$ is a covering family if and only if the sieve that it generates is a covering sieve for the given topology. It is easy to check that this defines a pretopology.

(PT 3) is sometimes replaced by a weaker axiom:

- (PT 3') (Identity) If $1_X : X \to X$ is the identity arrow, then $\{1_X\}$ is a covering family.

(PT 3) implies (PT 3'), but not conversely. However, suppose that we have a collection of covering families that satisfies (PT 0) through (PT 2) and (PT 3'), but not (PT 3). These families generate a pretopology. The topology generated by the original collection of covering families is then the same as the topology generated by the pretopology, because the sieve generated by an isomorphism $Y \to X$ is Hom(−, X). Consequently, if we restrict our attention to topologies, (PT 3) and (PT 3') are equivalent.

Sites and sheaves

Let C be a category and let J be a Grothendieck topology on C. The pair (C, J) is called a **site**.

A **presheaf** on a category is a contravariant functor from C to the category of all sets. Note that for this definition C is not required to have a topology. A sheaf on a site, however, should allow gluing, just like sheaves in classical topology. Consequently, we define a **sheaf** on a site to be a presheaf F such that for all objects X and all covering sieves S on X, the natural map Hom(Hom(−, X), F) → Hom(S, F) induced by the inclusion of S into Hom(−, X) is a bijection. Halfway in between a presheaf and a sheaf is the notion of a **separated presheaf**, where the natural map above is required to be only an injection, not a bijection, for all sieves S.

Using the Yoneda lemma, it is possible to show that a presheaf on the category $O(X)$ is a sheaf on the topology defined above if and only if it is a sheaf in the classical sense.

Sheaves on a pretopology have a particularly simple description: For each covering family $\{X_\alpha \to X\}$, the diagram

$$F(X) \to \prod_{\alpha \in A} F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta \in A} F(X_\alpha \times_X X_\beta)$$

must be an equalizer. For a separated presheaf, the first arrow need only be injective.

Similarly, one can define presheaves and sheaves of abelian groups, rings, modules, and so on. One can require either that a presheaf F is a contravariant functor to the category of abelian groups (or rings, or modules, etc.), or that F be an abelian group (ring, module, etc.) object in the category of all contravariant functors from C to the category of sets. These two definitions are equivalent.

Examples

The discrete and indiscrete topologies

Let **C** be any category. To define the **discrete topology**, we declare all sieves to be covering sieves. If **C** has all fibered products, this is equivalent to declaring all families to be covering families. To define the **indiscrete topology**, we declare only the sieves of the form Hom(−, X) to be covering sieves. The indiscrete topology is also known as the **biggest** or **chaotic** topology, and it is generated by the pretopology which has only isomorphisms for covering families. A sheaf on the indiscrete site is the same thing as a presheaf.

The canonical topology

Let **C** be any category. The Yoneda embedding gives a functor Hom(−, X) for each object X of **C**. The **canonical topology** is the biggest topology such that every representable presheaf Hom(−, X) is a sheaf. A covering sieve or covering family for this site is said to be *strictly universally epimorphic*. A topology which is less fine than the canonical topology, that is, for which every covering sieve is strictly universally epimorphic, is called **subcanonical**. Subcanonical sites are exactly the sites for which every presheaf of the form Hom(−, X) is a sheaf. Most sites encountered in practice are subcanonical.

Small site associated to a topological space

We repeat the example which we began with above. Let X be a topological space. We defined $O(X)$ to be the category whose objects are the open sets of X and whose morphisms are inclusions of open sets. The covering sieves on an object U of $O(X)$ were those sieves S satisfying the following condition:

- If W is the union of all the sets V such that $S(V)$ is non-empty, then $W = U$.

This topology can also naturally be expressed as a pretopology. We say that a family of inclusions $\{V_\alpha \subseteq U\}$ is a covering family if and only if the union $\cup V_\alpha$ equals U. This site is called the **small site associated to a topological space X.**

Big site associated to a topological space

Let *Spc* be the category of all topological spaces. Given any family of functions $\{u_\alpha : V_\alpha \to X\}$, we say that it is a **surjective family** or that the morphisms u_α are **jointly surjective** if $\cup u_\alpha(V_\alpha)$ equals X. We define a pretopology on *Spc* by taking the covering families to be surjective families all of whose members are open immersions. Let S be a sieve on *Spc*. S is a covering sieve for this topology if and only if:

- For all Y and every morphism $f : Y \to X$ in $S(Y)$, there exists a V and a $g : V \to X$ such that g is an open immersion, g is in $S(V)$, and f factors through g.
- If W is the union of all the sets $f(Y)$, where $f : Y \to X$ is in $S(Y)$, then $W = X$.

Fix a topological space X. Consider the comma category *Spc/X* of topological spaces with a fixed continuous map to X. The topology on *Spc* induces a topology on *Spc/X*. The covering sieves and covering families are almost exactly the same; the only difference is that now all the maps involved commute with the fixed maps to X. This is the **big site associated to a topological space** X . Notice that *Spc* is the big site associated to the one point space. This site was first considered by Jean Giraud.

The big and small sites of a manifold

Let M be a manifold. M has a category of open sets $O(M)$ because it is a topological space, and it gets a topology as in the above example. For two open sets U and V of M, the fiber product $U \times_M V$ is the open set $U \cap V$, which is still in $O(M)$. This means that the topology on $O(M)$ is defined by a pretopology, the same pretopology as before.

Let *Mfd* be the category of all manifolds and continuous maps. (Or smooth manifolds and smooth maps, or real analytic manifolds and analytic maps, etc.) *Mfd* is a subcategory of *Spc*, and open immersions are continuous (or smooth, or analytic, etc.), so *Mfd* inherits a topology from *Spc*. This lets us construct the big site of the manifold M as the site *Mfd/M*. We can also define this topology using the same pretopology we used above. Notice that to satisfy (PT 0), we need to check that for any continuous map of manifolds $X \to Y$ and any open subset U of Y, the fibered product $U \times_Y X$ is in *Mfd/M*. This is just the statement that the preimage of an open set is open. Notice, however, that not all fibered products exist in *Mfd* because the preimage of a smooth map at a critical value need not be a manifold.

Topologies and schemes

Fix a scheme X. There is more than one natural site associated to X. All of the following sites are subcanonical, and they are ordered from coarsest to finest.

The big and small Zariski sites

All schemes are topological spaces. We get the **small Zariski site of** X by considering X as a topological space and looking at the site $O(X)$. To define the big Zariski site, let *Zar* be the category whose objects are schemes and whose morphisms are morphisms of schemes. We define a pretopology on *Zar* by taking the covering families to be surjective families of scheme-theoretic open immersions. This defines a topology whose covering sieves S are characterized by the following two properties:

- For all Y and every morphism $f : Y \to X$ in $S(Y)$, there exists a V and a $g : V \to X$ such that g is an open immersion, g is in $S(V)$, and f factors through g.
- If W is the union of all the sets $f(Y)$, where $f : Y \to X$ is in $S(Y)$, then $W = X$.

Despite their outward similarities, the topology on *Zar* is *not* the restriction of the topology on *Spc*! This is because there are morphisms of schemes which are topologically open immersions but which are not scheme-theoretic open immersions. For example, let A be a non-reduced ring and let N be its ideal of nilpotents. The quotient map $A \to A/N$ induces a map Spec $A/N \to$ Spec A which is the identity on underlying topological spaces. To be a scheme-theoretic open immersion it must also induce an isomorphism on structure sheaves, which this map does not do. In fact, this map is a closed immersion.

We call *Zar/X* the **big Zariski site of** X .

The big and small Nisnevich sites

A family of morphisms $\{u_\alpha : X_\alpha \to X\}$ is a **Nisnevich cover** if the family is jointly surjective and each u_α is an étale morphism with the following additional property: For every point $x \in X$, there exists an α and a point $u \in X_\alpha$ such that the induced map of residue fields $k(x) \to k(u)$ is an isomorphism. (We call such an étale morphism with this property a *Nisnevich morphism*.) We define a pretopology on the category of schemes and morphisms of schemes by declaring covering families to be exactly the Nisnevich covers. This generates a topology called the **Nisnevich topology**. We write *Nis* for the category of schemes with the Nisnevich topology.

The **small Nisnevich site of** X is the category $O(X_{\mathrm{Nis}})$ whose objects are schemes U with a fixed Nisnevich morphism $U \to X$. The morphisms are morphisms of schemes compatible with the fixed maps to X. The **big Nisnevich site of** X is the category *Nis/X*, that is, the category of schemes with a fixed map to X, considered with the Nisnevich topology.

Nisnevich called this topology the *completely decomposed topology*. He introduced it in order to provide a cohomological interpretation of the class set of an affine group scheme (originally defined in adelic terms) and used it to partially prove the Grothendieck-Serre conjecture on rationally trivial torsors. This topology has also found important applications in → algebraic K-theory, A^1 homotopy theory and the theory of motives.

The big and small étale sites

We say that a family of morphisms $\{u_\alpha : X_\alpha \to X\}$ is an **étale cover** if the family is jointly surjective and each u_α is an étale morphism. We define a pretopology on the category of schemes and morphisms of schemes by declaring covering families to be exactly the étale covers. This generates a topology called the **étale topology**. We write *Ét* for the category of schemes with the étale topology.

The **small étale site of** X is the category $O(X_{ét})$ whose objects are schemes U with a fixed étale morphism $U \to X$. The morphisms are morphisms of schemes compatible with the fixed maps to X. The **big étale site of** X is the category *Ét/X*, that is, the category of schemes with a fixed map to X, considered with the étale topology.

We can define the étale topology using less data. First, we notice that the étale topology is finer than the Zariski topology. Consequently, to define an étale cover of a scheme X, it suffices to first cover X by open affine subschemes, that is, to take a Zariski cover, and then to define an étale cover of an affine scheme. We define an étale cover of an affine scheme X to be a surjective family $\{u_\alpha : X_\alpha \to X\}$ such that the set of all α is finite, each X_α is affine, and each u_α is étale. Then an étale cover of X is a family $\{u_\alpha : X_\alpha \to X\}$ which becomes an étale cover after base changing to any open affine subscheme of X.

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Serre spectral sequence

In mathematics, the **Serre spectral sequence** (sometimes **Leray-Serre spectral sequence** to acknowledge earlier work of Jean Leray in the Leray spectral sequence) is a basic tool of → algebraic topology. It expresses, in the language of → homological algebra the singular (co)homology of the total space E of a (Serre) fibration in terms of the (co)homology of the base space B and the fiber F. The result is due to → Jean-Pierre Serre in his doctoral dissertation (*Serre's thesis*).

Formulation

Let $f : E \to B$ be a Serre fibration of topological spaces, and let F be "the" fiber. The result is expressed by means of a spectral sequence and associated standard notation. Without simplifying assumptions, the notation has to be read correctly.

Cohomology spectral sequence

The Serre cohomology spectral sequence is the following:

$$E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E).$$

Here, at least under standard simplifying conditions, the coefficient group in the E_2-term is the q-th integral cohomology group of F, and the outer group is the singular cohomology of B with coefficients in that group.

Strictly speaking, what is meant is cohomology with respect to the local coefficient system on B given by the cohomology of the various fibers. Assuming for example, that B is simply connected, this collapses to the usual cohomology. For a path connected base, all the different fibers are homotopy equivalent. In particular, their cohomology is isomorphic, so the choice of "the" fiber does not give any ambiguity.

The abutment means integral cohomology of the total space.

There is a multiplicative structure

$$E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t},$$

coinciding on the E_2-term with qs-times the cup product, and with respect to which the differentials d_r are (graded) derivations inducing the product on the E_{r+1}-page from the one on the E_r-page.

Homology spectral sequence

Similarly to the cohomology spectral sequence, there is one for homology:

$$E^2_{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E),$$

where the notations are dual to the ones above.

It is actually a special case of a more general spectral sequence, namely the Serre spectral sequence for fibrations of simplicial sets. If f is a fibration of simplicial sets (a Kan fibration), such that $\pi_1(B)$, the first homotopy group of the simplicial set B, vanishes, there is a spectral sequence exactly as above. (Applying the functor which associates to any topological space its simplices to a fibration of topological spaces, one recovers the above sequence).

See also

- Gysin sequence
- Leray spectral sequence

References

The Serre spectral sequence is covered in most textbooks on algebraic topology, e.g.

- Allen Hatcher, *The Serre spectral sequence* [1]
- Edwin Spanier, *Algebraic topology*, Springer

The case of simplicial sets is treated in

- P. Goerss, R. Jardine, Simplicial homotopy theory, Birkhäuser

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[1] http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html

Sheaf

A **sheaf** is one of the large bundles in which cereal plants are bound after reaping. Accounts of two usages derived from this are found at:

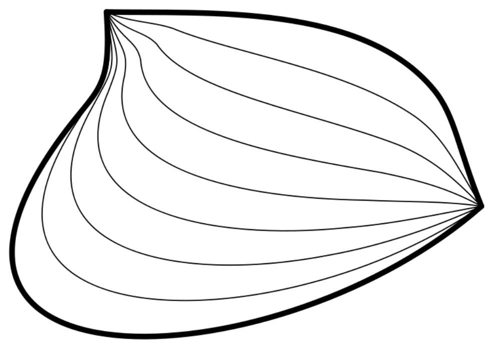
- Sheaf (mathematics)
- Sheaf toss (a Scottish sport)

See also

- River Sheaf
- Sceafa, a king of English legend

Homotopy theory

In → topology, two continuous functions from one topological space to another are called **homotopic** (Greek *homos* = identical and *topos* = place) if one can be "continuously deformed" into the other, such a deformation being called a **homotopy** between the two functions. An outstanding use of homotopy is the definition of homotopy groups and cohomotopy groups, important invariants in → algebraic topology.



The two bold paths shown above are homotopic relative to their endpoints. Thin lines mark isocontours of one possible homotopy.

In practice, there are technical difficulties in using homotopies with certain pathological spaces. Consequently most algebraic topologists work with compactly generated spaces, CW complexes, or spectra.

Formal definition

Formally, a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H: X \times [0,1] \to Y$ from the product of the space X with the unit interval [0,1] to Y such that, for all points x in X, $H(x,0)=f(x)$ and $H(x,1)=g(x)$.

A homotopy of a coffee cup into a doughnut (torus).

If we think of the second parameter of H as "time", then H describes a "continuous deformation" of f into g: at time 0 we have the function f, at time 1 we have the function g.

An alternative notation is to say that a homotopy between two continuous functions f , $g : X \to Y$ is a family of continuous functions $h_t: X \to Y$ for $t \in [0,1]$ such that $h_0 = f$ and $h_1 = g$ and the map $t \to h_t$ is continuous from [0,1] to the space of all continuous functions $X \to Y$. The two versions coincide by setting $h_t(x) = H(x,t)$.

Properties

Continuous functions f and g (both from topological space X to Y) are said to be homotopic if and only if there is a homotopy H taking f to g as described above. Being homotopic is an equivalence relation on the set of all continuous functions from X to Y. This homotopy relation is compatible with function composition in the following sense: if f_1, $g_1: X \to Y$ are homotopic, and f_2, $g_2: Y \to Z$ are homotopic, then their compositions $f_2 \circ f_1$ and $g_2 \circ g_1: X \to Z$ are homotopic as well.

Homotopy equivalence and null-homotopy

Given two spaces X and Y, we say they are **homotopy equivalent** or of the same **homotopy type** if there exist continuous maps f: $X \to Y$ and g: $Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

The maps f and g are called **homotopy equivalences** in this case. Clearly, every homeomorphism is a homotopy equivalence, but the converse is not true: for example, a solid disk is not homeomorphic to a single point, although the disk and the point are homotopy equivalent.

Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. For example, a solid disk or solid ball is homotopy equivalent to a point, and $\mathbf{R}^2 - \{(0,0)\}$ is homotopy equivalent to the unit circle S^1. Spaces that are homotopy equivalent to a point are called contractible.

A function f is said to be **null-homotopic** if it is homotopic to a constant function. (The homotopy from f to a constant function is then sometimes called a **null-homotopy**.) For example, a map from the circle S^1 is null-homotopic precisely when it can be extended to a map of the disc D^2.

It follows from these definitions that a space X is contractible if and only if the identity map from X to itself—which is always a homotopy equivalence—is null-homotopic.

Homotopy invariance

Homotopy equivalence is important because in → algebraic topology many concepts are **homotopy invariant**, that is, they respect the relation of homotopy equivalence. For example, if X and Y are homotopy equivalent spaces, then:

- if X is path-connected, then so is Y
- if X is simply connected, then so is Y
- the (singular) homology and cohomology groups of X and Y are isomorphic
- if X and Y are path-connected, then the → fundamental groups of X and Y are isomorphic, and so are the higher homotopy groups. Without the path-connectedness assumption, one has $\pi_1(X, x_0)$ isomorphic to $\pi_1(Y, f(x_0))$ where f: $X \to Y$ is a homotopy equivalence and x_0 a given point in X.

An example of an algebraic invariant of topological spaces which is not homotopy-invariant is compactly supported homology (which is, roughly speaking, the homology of the compactification, and compactification is not homotopy-invariant).

Relative homotopy

In order to define the → fundamental group, one needs the notion of **homotopy relative to a subspace**. These are homotopies which keep the elements of the subspace fixed. Formally: if f and g are continuous maps from X to Y and K is a subset of X, then we say that f and g are homotopic relative to K if there exists a homotopy H: $X \times [0,1] \to Y$ between f and g such that $H(k,t) = f(k) = g(k)$ for all $k \in K$ and $t \in [0,1]$. Also, if g is a retract from X to K and f is the identity map, this is known as a strong deformation retract of X to K. When K is a point, the term **pointed homotopy** is used.

Homotopy groups

Since the relation of two functions $f, g : X \to Y$ being homotopic relative to a subspace is an equivalence relation, we can look at the equivalence classes of maps between a fixed X and Y. If we fix $X = [0,1]^n$, the unit interval [0,1] crossed with itself n times, and we take our subspace to be its boundary $\partial([0,1]^n)$ then the equivalence classes form a group, denoted $\pi_n(Y, y_0)$, where y_0 is the image of the subspace $\partial([0,1]^n)$.

We can define the action of one equivalence class on another, and so we get a group. These groups are called the homotopy groups. In the case $n = 1$, it is also called the → fundamental group.

Homotopy category

The idea of homotopy can be turned into a formal category of category theory. The **homotopy category** is the category whose objects are topological spaces, and whose morphisms are homotopy equivalence classes of continuous maps. Two topological spaces X and Y are isomorphic in this category if and only if they are homotopy-equivalent. Then a functor on the category of topological spaces is homotopy invariant if it can be expressed as a functor on the homotopy category.

For example, homology groups are a *functorial* homotopy invariant: this means that if f and g from X to Y are homotopic, then the group homomorphisms induced by f and g on the level of homology groups are the same: $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$ for all n. Likewise, if X and Y are in addition path-connected, then the group homomorphisms induced by f and g on the level of homotopy groups are also the same: $\pi_n(f) = \pi_n(g) : \pi_n(X) \to \pi_n(Y)$.

Timelike homotopy

On a Lorentzian manifold, certain curves are distinguished as timelike. A timelike homotopy between two timelike curves is a homotopy such that each intermediate curve is timelike. No closed timelike curve (CTC) on a Lorentzian manifold is timelike homotopic to a point (that is, null timelike homotopic); such a manifold is therefore said to be multiply connected by timelike curves. A manifold such as the 3-sphere can be simply connected (by any type of curve), and yet be multiply timelike connected.

Homotopy lifting property

If we have a homotopy $H: X \times [0,1] \to Y$ and a cover $p: \tilde{Y} \to Y$ and we are given a map $\tilde{h}_0: X \to \tilde{Y}$ such that $\tilde{h}_0 = p \circ h_0$ ($\tilde{h}_0$ is called a lift of h_0), then we can lift all of H to a map $\tilde{H}: X \times [0,1] \to \tilde{Y}$ such that $p \circ \tilde{H} = H$. The homotopy lifting property is used to characterize fibrations.

Homotopy extension property

Another useful property involving homotopy is the homotopy extension property, which characterizes the extension of a homotopy between two functions from a subset of some set to the set itself. It is useful when dealing with cofibrations.

Isotopy

In case the two given continuous functions *f* and *g* from the topological space *X* to the topological space *Y* are homeomorphisms, one can ask whether they can be connected 'through homeomorphisms'. This gives rise to the concept of **isotopy**, which is a homotopy, *H*, in the notation used before, such that for each fixed *t*, $H(x,t)$ gives a homeomorphism.

Requiring that two homeomorphisms be isotopic really is a stronger requirement than that they be homotopic. For example, the map of the unit disc in R^2 defined by $f(x,y) = (-x, -y)$ is equivalent to a 180-degree rotation around the origin, and so the identity map and *f* are isotopic because they can be connected by rotations. However, the map on the interval $[-1,1]$ in *R* defined by $f(x) = -x$ is *not* isotopic to the identity. Loosely speaking, any homotopy from *f* to the identity would have to exchange the endpoints, which would mean that they would have to 'pass through' each other. Moreover, *f* has changed the orientation of the interval, hence it cannot be isotopic to the identity. However, the maps are homotopic; one homotopy from *f* to the identity is H: $[-1,1] \times [0,1] \to [-1,1]$ given by $H(x,y) = 2yx\text{-}x$.

In geometric topology—for example in knot theory—the idea of isotopy is used to construct equivalence relations. For example, when should two knots be considered the same? We take two knots, K_1 and K_2, in three-dimensional space. A knot is an embedding of a one-dimensional space, the "loop of string", into this space, and an embedding is simply a homeomorphism. The intuitive idea of *deforming* one to the other should correspond to a path of embeddings: a continuous function starting at t=0 with the K_1 embedding, ending at t=1 with the K_2 embedding, with all intermediate values being embeddings; this corresponds to the definition of isotopy. An ambient isotopy, studied in this context, is an isotopy of the larger space, considered in light of its action on the embedded submanifold.

Applications

Based on the concept of the homotopy, computation methods for algebraic and differential equations are developed. The methods for algebraic equations include the homotopy continuation method and the continuation method. The methods for differential equations include the homotopy analysis method.

See also

- Mapping class group
- Homeotopy
- Regular homotopy
- Poincaré conjecture
- Homotopy analysis method

Fundamental group

In mathematics, more specifically → algebraic topology, the **fundamental group** or **Poincaré group** is a group associated to any given pointed topological space that provides a way of determining when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. Intuitively, it records information about the basic shape, or *holes*, of the topological space. The fundamental group is the first and simplest of the homotopy groups.

Fundamental groups can be studied using the theory of covering spaces, since a fundamental group coincides with the group of deck transformations of the associated universal covering space. Its abelianisation can be identified with the first homology group of the space. When the topological space is homeomorphic to a simplicial complex, its fundamental group can be described explicitly in terms of generators and relations.

Historically, the concept of fundamental group first emerged in the theory of Riemann surfaces, in the work of Bernhard Riemann, Henri Poincaré and Felix Klein, where it describes the monodromy properties of complex functions, as well as providing a complete topological classification of closed surfaces.

Definition

Let X be an arcwise-connected topological space.

Define a **path** as a continuous function f from $[0,1] \to X$.

Define a **loop** as a closed path, i.e. a loop is a path f with $f(0) = f(1)$.

Define an equivalence relation between two loops f and g by calling them equivalent iff there is a continuous function F $[0,1]\times[0,1] \to X\times X$ s.t. $F(0, \mathrm{p}) = f(p)$ and $F(1, p) = g(p)$ for all p in $[0, 1]$.

Define **inverse** of a loop f as a new loop $-f$, s.t. $(-f)(x) = f(1 - x)$.

For two paths f and g s.t. $f(1) = g(0)$, define **addition** as a new path $f + g$, s.t.

$$(f+g)(x) = \begin{cases} f(2x) \text{ if } 0 \leq x \leq 1/2 \\ g(2(x-1/2)) \text{ if } 1/2 < x \leq 1 \end{cases}$$

To define **addition of two loops** f and g, choose a path h from $f(0)$ to $g(0)$. Then define $f * g = f + h + g + (-h)$.

In other words, to add loops f and g, create a new loop by tracing f first, then tracing a path to starting point of g, then g, then a path back to starting point of f.

Then the group of all such equivalence classes (i.e. loops in X modulo the equivalence defined above), form a group with respect to loop additions. This group is called the **fundamental group** of the space X.

Intuition

Start with a space (e.g. a surface), and some point in it, and all the loops both starting and ending at this point — paths that start at this point, wander around and eventually return to the starting point. Two loops can be combined together in an obvious way: travel along the first loop, then along the second. Two loops are considered equivalent if one can be deformed into the other without breaking. The set of all such loops with this method of combining and this equivalence between them is the fundamental group.

For the precise definition, let X be a topological space, and let x_0 be a point of X. We are interested in the set of continuous functions $f : [0,1] \to X$ with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with **base point** x_0. Any two such loops, say f and g, are considered equivalent if there is a continuous function $h : [0,1] \times [0,1] \to X$ with the property that, for all $0 \leq t \leq 1$, $h(t, 0) = f(t)$, $h(t, 1) = g(t)$ and $h(0, t) = x_0 = h(1, t)$. Such an h is called a **homotopy** from f to g, and the corresponding equivalence classes are called **homotopy classes**.

The product $f * g$ of two loops f and g is defined by setting $(f * g)(t) := f(2t)$ if $0 \le t \le 1/2$ and $(f * g)(t) := g(2t - 1)$ if $1/2 \le t \le 1$. Thus the loop $f * g$ first follows the loop f with "twice the speed" and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f * g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The identity element is the constant map at the basepoint, and the inverse of a loop f is the loop g defined by $g(t) = f(1 - t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference so long as the space X is path-connected. For path-connected spaces, therefore, we can write $\pi_1(X)$ instead of $\pi_1(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

Examples

Trivial fundamental group. In Euclidean space $\mathbf{R}^n$, or any convex subset of $\mathbf{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore the trivial group with one element. A path-connected space with a trivial fundamental group is said to be simply connected.

Infinite cyclic fundamental group. The circle. Each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m + n$ times. So the fundamental group of the circle is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers. This fact can be used to give proofs of the → Brouwer fixed point theorem and the Borsuk–Ulam theorem in dimension 2.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

Free groups of higher rank: Graphs. Unlike the homology groups and higher homotopy groups associated to a topological space, the fundamental group need not be abelian. For example, the fundamental group of the figure eight is the free group on two letters. More generally, the fundamental group of any graph G is a free group. Here the rank of the free group is equal to $1 - \chi(G)$: one minus the Euler characteristic of G, when G is connected.

Knot theory. A somewhat more sophisticated example of a space with a non-abelian fundamental group is the complement of a trefoil knot in $\mathbf{R}^3$.

Functoriality

If $f : X \to Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0. This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the induced homomorphism, is written as $\pi(f)$ or, more commonly,

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

We thus obtain a functor from the category of topological spaces with base point to the category of groups.

It turns out that this functor cannot distinguish maps which are homotopic relative to the base point: if f and $g : X \to Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, and f and g are homotopic relative to $\{x_0\}$, then $f_* = g_*$. As a consequence, two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

The fundamental group functor takes products to products and coproducts to coproducts. That is, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

(In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and * the free product of groups.) Both formulas generalize to arbitrary products. Furthermore the latter formula is a special case of the Seifert–van Kampen theorem which states that the fundamental group functor takes pushouts along inclusions to pushouts.

Fibrations

A generalization of a product of spaces is given by a fibration,

$$F \to E \to B.$$

Here the total space *E* is a sort of "twisted product" of the base space *B* and the fiber *F*. In general the fundamental groups of *B*, *E* and *F* are terms in a long exact sequence involving higher homotopy groups. When all the spaces are connected, this has the following consequences for the fundamental groups:

- $\pi_1(B)$ and $\pi_1(E)$ are isomorphic if *F* is simply connected
- $\pi_{n+1}(B)$ and $\pi_n(F)$ are isomorphic if *E* is contractible

The latter is often applied to the situation *E* = path space of *B*, *F* = loop space of *B* or *B* = classifying space *BG* of a topological group *G*, *B* = universal *G*-bundle *EG*.

Relationship to first homology group

The fundamental groups of a topological space *X* are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism from the fundamental group $\pi_1(X, x_0)$ to the homology group $H_1(X)$. If *X* is path-connected, then this homomorphism is surjective and its kernel is the commutator subgroup of $\pi_1(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi_1(X, x_0)$. This is a special case of the → Hurewicz theorem of algebraic topology.

Universal covering space

If *X* is a topological space that is path connected, locally path connected and locally simply connected, then it has a simply connected universal covering space on which the fundamental group $\pi(X,x_0)$ acts freely by deck transformations with quotient space *X*. This space can be constructed analogously to the fundamental group by taking pairs (x, γ), where *x* is a point in *X* and γ is a homotopy class of paths from x_0 to *x* and the action of $\pi(X, x_0)$ is by concatenation of paths. It is uniquely determined as a covering space.

Examples

Let *G* be a connected, simply connected compact Lie group, for example the special unitary group SU_n, and let Γ be a finite subgroup of *G*. Then the homogeneous space $X = G/\Gamma$ has fundamental group Γ, which acts by right multiplication on the universal covering space *G*. Among the many variants of this construction, one of the most important is given by locally symmetric spaces $X = \Gamma\backslash G/K$, where

- *G* is a non-compact simply connected, connected Lie group (often semisimple),
- *K* is a maximal compact subgroup of *G*
- Γ is a discrete countable torsion-free subgroup of *G*.

In this case the fundamental group is Γ and the universal covering space *G*/*K* is actually contractible (by the Cartan decomposition for Lie groups).

As an example take $G = SL_2(\mathbf{R})$, $K = SO_2$ and Γ any torsion-free congruence subgroup of the modular group $SL_2(\mathbf{Z})$.

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The edge-path group is naturally isomorphic to $\pi_1(|X|, v)$, the fundamental group of the geometric realisation $|X|$ of X. Since it depends only on the 2-skeleton X^2 of X (i.e. the vertices, edges and triangles of X), the groups $\pi_1(|X|,v)$ and $\pi_1(|X^2|, v)$ are isomorphic.

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It is well-known that this method can also be used to compute the fundamental group of an arbitrary topological space. This was doubtless known to Čech and Leray and explicitly appeared as a remark in a paper by Weil (1960); various other authors such as L. Calabi, W-T. Wu and N. Berikashvili have also published proofs. In the simplest case of a compact space X with a finite open covering in which all non-empty finite intersections of open sets in the covering are contractible, the fundamental group can be identified with the edge-path group of the simplicial complex corresponding to the nerve of the covering.

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Every group can be realized as the fundamental group of a connected CW-complex of dimension 2 (or higher). As noted above, though, only free groups can occur as fundamental groups of 1-dimensional CW-complexes (that is, graphs).

Every finitely presented group can be realized as the fundamental group of a compact, connected, smooth manifold of dimension 4 (or higher). But there are severe restrictions on which groups occur as fundamental groups of low-dimensional manifolds. For example, no free abelian group of rank 4 or higher can be realized as the fundamental group of a manifold of dimension 3 or less.

Related concepts

The fundamental group measures the 1-dimensional hole structure of a space. For studying "higher-dimensional holes", the homotopy groups are used. The elements of the n-th homotopy group of X are homotopy classes of (basepoint-preserving) maps from S^n to X.

The set of loops at a particular base point can be studied without regarding homotopic loops as equivalent. This larger object is the loop space.

For topological groups, a different group multiplication may be assigned to the set of loops in the space, with pointwise multiplication rather than concatenation. The resulting group is the loop group.

Fundamental groupoid

Rather than singling out one point and considering the loops based at that point up to homotopy, one can also consider *all* paths in the space up to homotopy (fixing the initial and final point). This yields not a group but a → groupoid, the **fundamental groupoid** of the space.

See also

- Homotopy group, generalization of fundamental group

There are also similar notions of fundamental group for algebraic varieties (the étale fundamental group) and for orbifolds (the orbifold fundamental group).

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Fundamental groupoid

In mathematics, more specifically → algebraic topology, the **fundamental group** or **Poincaré group** is a group associated to any given pointed topological space that provides a way of determining when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. Intuitively, it records information about the basic shape, or *holes*, of the topological space. The fundamental group is the first and simplest of the homotopy groups.

Fundamental groups can be studied using the theory of covering spaces, since a fundamental group coincides with the group of deck transformations of the associated universal covering space. Its abelianisation can be identified with the first homology group of the space. When the topological space is homeomorphic to a simplicial complex, its fundamental group can be described explicitly in terms of generators and relations.

Historically, the concept of fundamental group first emerged in the theory of Riemann surfaces, in the work of Bernhard Riemann, Henri Poincaré and Felix Klein, where it describes the monodromy properties of complex functions, as well as providing a complete topological classification of closed surfaces.

Definition

Let X be an arcwise-connected topological space.

Define a **path** as a continuous function f from $[0,1] \to X$.

Define a **loop** as a closed path, i.e. a loop is a path f with $f(0) = f(1)$.

Define an equivalence relation between two loops f and g by calling them equivalent iff there is a continuous function F $[0,1]\times[0,1] \to X\times X$ s.t. $F(0, \mathrm{p}) = f(p)$ and $F(1, p) = g(p)$ for all p in [0, 1].

Define **inverse** of a loop f as a new loop $-f$, s.t. $(-f)(x) = f(1 - x)$.

For two paths f and g s.t. $f(1) = g(0)$, define **addition** as a new path $f + g$, s.t.

$$(f+g)(x) = \begin{cases} f(2x) \text{ if } 0 \le x \le 1/2 \\ g(2(x-1/2)) \text{ if } 1/2 < x \le 1 \end{cases}$$

To define **addition of two loops** f and g, choose a path h from $f(0)$ to $g(0)$. Then define $f * g = f + h + g + (-h)$.

In other words, to add loops f and g, create a new loop by tracing f first, then tracing a path to starting point of g, then g, then a path back to starting point of f.

Then the group of all such equivalence classes (i.e. loops in X modulo the equivalence defined above), form a group with respect to loop additions. This group is called the **fundamental group** of the space X.

Intuition

Start with a space (e.g. a surface), and some point in it, and all the loops both starting and ending at this point — paths that start at this point, wander around and eventually return to the starting point. Two loops can be combined together in an obvious way: travel along the first loop, then along the second. Two loops are considered equivalent if one can be deformed into the other without breaking. The set of all such loops with this method of combining and this equivalence between them is the fundamental group.

For the precise definition, let X be a topological space, and let x_0 be a point of X. We are interested in the set of continuous functions $f : [0,1] \to X$ with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with **base point** x_0. Any two such loops, say f and g, are considered equivalent if there is a continuous function $h : [0,1] \times [0,1] \to X$ with the property that, for all $0 \le t \le 1$, $h(t, 0) = f(t)$, $h(t, 1) = g(t)$ and $h(0, t) = x_0 = h(1, t)$. Such an h is called a **homotopy** from f to g, and the corresponding equivalence classes are called **homotopy classes**.

The product $f * g$ of two loops f and g is defined by setting $(f * g)(t) := f(2t)$ if $0 \le t \le 1/2$ and $(f * g)(t) := g(2t - 1)$ if $1/2 \le t \le 1$. Thus the loop $f * g$ first follows the loop f with "twice the speed" and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f * g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The identity element is the constant map at the basepoint, and the inverse of a loop f is the loop g defined by $g(t) = f(1 - t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference so long as the space X is path-connected. For path-connected spaces, therefore, we can write $\pi_1(X)$ instead of $\pi_1(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

Examples

Trivial fundamental group. In Euclidean space $\mathbf{R}^n$, or any convex subset of $\mathbf{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore the trivial group with one element. A path-connected space with a trivial fundamental group is said to be simply connected.

Infinite cyclic fundamental group. The circle. Each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m + n$ times. So the fundamental group of the circle is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers. This fact can be used to give proofs of the → Brouwer fixed point theorem and the Borsuk–Ulam theorem in dimension 2.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

Free groups of higher rank: Graphs. Unlike the homology groups and higher homotopy groups associated to a topological space, the fundamental group need not be abelian. For example, the fundamental group of the figure eight is the free group on two letters. More generally, the fundamental group of any graph G is a free group. Here the rank of the free group is equal to $1 - \chi(G)$: one minus the Euler characteristic of G, when G is connected.

Knot theory. A somewhat more sophisticated example of a space with a non-abelian fundamental group is the complement of a trefoil knot in $\mathbf{R}^3$.

Functoriality

If $f : X \to Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0. This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the induced homomorphism, is written as $\pi(f)$ or, more commonly,

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

We thus obtain a functor from the category of topological spaces with base point to the category of groups.

It turns out that this functor cannot distinguish maps which are homotopic relative to the base point: if f and $g : X \to Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, and f and g are homotopic relative to $\{x_0\}$, then $f_* = g_*$. As a consequence, two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

The fundamental group functor takes products to products and coproducts to coproducts. That is, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

(In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and * the free product of groups.) Both formulas generalize to arbitrary products. Furthermore the latter formula is a special case of the Seifert–van Kampen theorem which states that the fundamental group functor takes pushouts along inclusions to pushouts.

Fibrations

A generalization of a product of spaces is given by a fibration,

$$F \to E \to B.$$

Here the total space E is a sort of "twisted product" of the base space B and the fiber F. In general the fundamental groups of B, E and F are terms in a long exact sequence involving higher homotopy groups. When all the spaces are connected, this has the following consequences for the fundamental groups:

- $\pi_1(B)$ and $\pi_1(E)$ are isomorphic if F is simply connected
- $\pi_{n+1}(B)$ and $\pi_n(F)$ are isomorphic if E is contractible

The latter is often applied to the situation E = path space of B, F = loop space of B or B = classifying space BG of a topological group G, B = universal G-bundle EG.

Relationship to first homology group

The fundamental groups of a topological space X are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism from the fundamental group $\pi_1(X, x_0)$ to the homology group $H_1(X)$. If X is path-connected, then this homomorphism is surjective and its kernel is the commutator subgroup of $\pi_1(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi_1(X, x_0)$. This is a special case of the → Hurewicz theorem of algebraic topology.

Universal covering space

If X is a topological space that is path connected, locally path connected and locally simply connected, then it has a simply connected universal covering space on which the fundamental group $\pi(X,x_0)$ acts freely by deck transformations with quotient space X. This space can be constructed analogously to the fundamental group by taking pairs (x, γ), where x is a point in X and γ is a homotopy class of paths from x_0 to x and the action of $\pi(X, x_0)$ is by concatenation of paths. It is uniquely determined as a covering space.

Examples

Let G be a connected, simply connected compact Lie group, for example the special unitary group SU_n, and let Γ be a finite subgroup of G. Then the homogeneous space $X = G/\Gamma$ has fundamental group Γ, which acts by right multiplication on the universal covering space G. Among the many variants of this construction, one of the most important is given by locally symmetric spaces $X = \Gamma\backslash G/K$, where

- G is a non-compact simply connected, connected Lie group (often semisimple),
- K is a maximal compact subgroup of G
- Γ is a discrete countable torsion-free subgroup of G.

In this case the fundamental group is Γ and the universal covering space G/K is actually contractible (by the Cartan decomposition for Lie groups).

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Homology theory

In mathematics, **homology theory** is the axiomatic study of the intuitive geometric idea of *homology of cycles* on topological spaces. It can be broadly defined as the study of homology theories on topological spaces.

Simple explanation

At the intuitive level *homology* is taken to be an equivalence relation, such that chains *C* and *D* are *homologous* on the space *X* if the chain *C* − *D* is a *boundary* of a chain of one dimension higher. The simplest case is in graph theory, with *C* and *D* vertices and homology with a meaning coming from the oriented edge *E* from *P* to *Q* having boundary *Q* — *P*. A collection of edges from *D* to *C*, each one joining up to the one before, is a homology. In general, a *k*-chain is thought of as a formal combination

$$\sum a_i d_i$$

where the a_i are integers and the d_i are *k*-dimensional simplices on *X*. The boundary concept here is that taken over from the boundary of a simplex; it allows a high-dimensional concept which for *k* = 1 is the kind of telescopic cancellation seen in the graph theory case. This explanation is in the style of 1900, and proved somewhat naive, technically speaking.

Example of a torus surface

For example if X is a 2-torus T, a one-dimensional *cycle* on T is in intuitive terms a linear combination of curves drawn on T, which closes up on itself (cycle condition, equivalent to having no net boundary). If C and D are cycles each wrapping once round T in the same way, we can find explicitly an oriented area on T with boundary $C - D$. One can prove that the homology classes of 1-cycles with integer coefficients form a free abelian group with two generators, one generator for each of the two different ways round the 'doughnut'.

The nineteenth century

This level of understanding was common property in the mathematics of the nineteenth century, starting with the idea of Riemann surface. At the end of the century, the work of Poincaré had provided a much more general, though still intuitively-based, setting.

For example, it is considered that the general Stokes' theorem was first stated in 1899 by Poincaré: it involves necessarily both an *integrand* (we would now say, a differential form), and a region of integration (a p-chain), with two kinds of *boundary* operators, one of which in modern terms is the exterior derivative, and the other a geometric boundary operator on chains that includes *orientation* and can be used for homology theory. The two *boundaries* appear as adjoint operators, with respect to integration.

Twentieth century beginnings

Rather loose, geometric arguments with homology were only gradually replaced at the beginning of the twentieth century by rigorous techniques. To begin with, the style of the era was to use combinatorial topology (the fore-runner of today's → algebraic topology). That assumes that the spaces treated are simplicial complexes, while the most interesting spaces are usually manifolds, so that artificial triangulations have to be introduced to apply the tools. Pioneers such as Solomon Lefschetz and Marston Morse still preferred a geometric approach. The combinatorial stance did allow Brouwer to prove foundational results such as the simplicial approximation theorem, at the base of the idea that homology is a functor (as it would later be put). Brouwer was able to prove the Jordan curve theorem, basic for complex analysis, and the invariance of domain, using the new tools; and remove the suspicion attached to topological arguments as handwaving.

Towards algebraic topology

The transition to *algebraic* topology is usually attributed to the influence of Emmy Noether, who insisted that homology classes lay in quotient groups — a point of view now so fundamental that it is taken as a definition.[1] In fact Noether in the period from 1920 onwards was with her students elaborating the theory of modules for any ring, giving rise when the two ideas were combined to the concept of *homology with coefficients in a ring*. Before that, coefficients (that is, the sense in which chains are linear combinations of the basic geometric chains traced on the space) had usually been integers, real or complex numbers, or sometimes residue classes mod 2. In the new setting, there would be no reason not to take residues mod 3, for example: to be a cycle is then a more complex geometric condition, exemplified in graph theory terms by having the number of incoming edges at every vertex a multiple of 3. But in algebraic terms, the definitions present no new problem. The → universal coefficient theorem explains that homology *with integer coefficients* determines all other homology theories, by use of the tensor product; it is not anodyne, in that (as we would now put it) the tensor product has derived functors that enter into a general formulation.

Cohomology, and singular homology

The 1930s were the decade of the development of → cohomology theory, as several research directions grew together and the De Rham cohomology that was implicit in Poincaré's work cited earlier became the subject of definite theorems. Cohomology and homology are *dual* theories, in a sense that required detailed working out; at the same time it was realised that homology, that was, simplicial homology, was far from being at the end of its story. The definition of singular homology avoided the need for the apparatus of triangulations, at a cost of moving to infinitely-generated modules.

Axiomatics and extraordinary theories

The development of algebraic topology from 1940 to 1960 was very rapid, and the role of homology theory was often as 'baseline' theory, easy to compute and in terms of which topologists sought to calculate with other functors. The axiomatisation of homology theory by Eilenberg and Steenrod (the Eilenberg-Steenrod axioms) revealed that what various candidate homology *theories* had in common was, roughly speaking, some exact sequences and in particular the Mayer–Vietoris sequence, and the *dimension axiom* that calculated the homology of the point. The dimension axiom was relaxed to admit the (co)homology derived from topological → K-theory, and cobordism theory, in a vast extension to the **extraordinary (co)homology theories** that became standard in → homotopy theory. These can be easily characterised for the category of CW complexes.

- List of cohomology theories

Current state of homology theory

For more general (i.e. less well-behaved) spaces, recourse to ideas from sheaf theory brought some extension of homology theories, particularly the Borel-Moore homology for locally compact spaces.

The basic chain complex apparatus of homology theory has long since become a separate piece of technique in → homological algebra, and has been applied independently, for example to group cohomology. Therefore there is no longer one homology theory, but many homology and cohomology theories in mathematics.

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Homological algebra

Homological algebra is the branch of mathematics which studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to → algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert.

The development of homological algebra was closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. The hidden fabric of mathematics is woven of **chain complexes**, which manifest themselves through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. A powerful tool for doing this is provided by spectral sequences.

From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. → K-theory is an independent discipline which draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes.

Chain complexes and homology

The **chain complex** is the central notion of homological algebra. It is a sequence $(C_\bullet, d_\bullet)$ of abelian groups and group homomorphisms, with the property that the composition of any two consecutive maps is zero:

$$C_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots, \quad d_n \circ d_{n+1} = 0.$$

The elements of C_n are called n-**chains** and the homomorphisms d_n are called the **boundary maps** or **differentials**. The **chain groups** C_n may be endowed with extra structure; for example, they may be vector spaces or modules over a fixed ring R. The differentials must preserve the extra structure if it exists; for example, they must be linear maps or homomorphisms of R-modules. For notational convenience, restrict attention to abelian groups (more correctly, to the category **Ab** of abelian groups); a celebrated theorem by Barry Mitchell implies the results will generalize to any abelian category. Every chain complex defines two further sequences of abelian groups, the **cycles** $Z_n = \operatorname{Ker} d_n$ and the **boundaries** $B_n = \operatorname{Im} d_{n+1}$, where Ker d and Im d denote the kernel and the image of d. Since the composition of two consecutive boundary maps is zero, these groups are embedded into each other as

$$B_n \subseteq Z_n \subseteq C_n.$$

Subgroups of abelian groups are automatically normal; therefore we can define the nth **homology group** $H_n(C)$ as the factor group of the n-cycles by the n-boundaries,

$$H_n(C) = Z_n/B_n = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

A chain complex is called **acyclic** or an **exact sequence** if all its homology groups are zero.

Chain complexes arise in abundance in algebra and → algebraic topology. For example, if X is a topological space then the singular chains $C_n(X)$ are formal linear combinations of continuous maps from the standard n-simplex into X; if K is a simplicial complex then the simplicial chains $C_n(K)$ are formal linear combinations of the n-simplices of X; if $A = F/R$ is a presentation of an abelian group A by generators and relations, where F is a free abelian group spanned by the generators and R is the subgroup of relations, then letting $C_1(A) = R$, $C_0(A) = F$, and $C_n(A) = 0$ for all other n defines a sequence of abelian groups. In all these cases, there are natural differentials d_n making C_n into a chain complex, whose homology reflects the structure of the topological space X, the simplicial complex K, or the abelian group A. In the case of topological spaces, we arrive at the notion of singular homology, which plays a fundamental role in investigating the properties of such spaces, for example, manifolds.

On a philosophical level, homological algebra teaches us that certain chain complexes associated with algebraic or geometric objects (topological spaces, simplicial complexes, *R*-modules) contain a lot of valuable algebraic information about them, with the homology being only the most readily available part. On a technical level, homological algebra provides the tools for manipulating complexes and extracting this information. Here are two general illustrations.

- Two objects *X* and *Y* are connected by a map *f* between them. Homological algebra studies the relation, induced by the map *f*, between chain complexes associated to *X* and *Y* and their homology. This is generalized to the case of several objects and maps connecting them. Phrased in the language of category theory, homological algebra studies the functorial properties of various constructions of chain complexes and of the homology of these complexes.
- An object *X* admits multiple descriptions (for example, as a topological space and as a simplicial complex) or the complex $C_\bullet(X)$ is constructed using some 'presentation' of *X*, which involves non-canonical choices. It is important to know the effect of change in the description of *X* on chain complexes associated to *X*. Typically, the complex and its homology $H_\bullet(C)$ are functorial with respect to the presentation; and the homology (although not the complex itself) is actually independent of the presentation chosen, thus it is an invariant of *X*.

Functoriality

A continuous map of topological spaces gives rise to a homomorphism between their *n*th homology groups for all *n*. This basic fact of → algebraic topology finds a natural explanation through certain properties of chain complexes. Since it is very common to study several topological spaces simultaneously, in homological algebra one is led to simultaneous consideration of multiple chain complexes.

A **morphism** between two chain complexes, $F : C_\bullet \to D_\bullet$, is a family of homomorphisms of abelian groups $F_n : C_n \to D_n$ that commute with the differentials, in the sense that $F_{n-1} \bullet d_n^C = d_n^D \bullet F_n$ for all *n*. A morphism of chain complexes induces a morphism $H_\bullet(F)$ of their homology groups, consisting of the homomorphisms $H_n(F)$: $H_n(C) \to H_n(D)$ for all *n*. A morphism *F* is called a **quasi-isomorphism** if it induces an isomorphism on the *n*th homology for all *n*.

Many constructions of chain complexes arising in algebra and geometry, including singular homology, have the following functoriality property: if two objects *X* and *Y* are connected by a map *f*, then the associated chain complexes are connected by a morphism $F = C(f)$ from $C_\bullet(X)$ to $C_\bullet(Y)$, and moreover, the composition $g \bullet f$ of maps $f: X \to Y$ and $g: Y \to Z$ induces the morphism $C(g \bullet f)$ from $C_\bullet(X)$ to $C_\bullet(Z)$ that coincides with the composition $C(g) \bullet C(f)$. It follows that the homology groups $H_\bullet(C)$ are functorial as well, so that morphisms between algebraic or topological objects give rise to compatible maps between their homology.

The following definition arises from a typical situation in algebra and topology. A triple consisting of three chain complexes $L_\bullet, M_\bullet, N_\bullet$ and two morphisms between them, $f : L_\bullet \to M_\bullet, g : M_\bullet \to N_\bullet$, is called an **exact triple**, or a **short exact sequence of complexes**, and written as

$$0 \longrightarrow L_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} N_\bullet \longrightarrow 0,$$

if for any *n*, the sequence

$$0 \longrightarrow L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \longrightarrow 0$$

is a short exact sequence of abelian groups. By definition, this means that f_n is an injection, g_n is a surjection, and Im f_n = Ker g_n. One of the most basic theorems of homological algebra, sometimes known as the zig-zag lemma, states that, in this case, there is a **long exact sequence in homology**

$$\dots \longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(M) \xrightarrow{H_n(g)} H_n(N) \xrightarrow{\delta_n} H_{n-1}(L) \xrightarrow{H_{n-1}(f)} H_{n-1}(M) \longrightarrow \dots,$$

where the homology groups of *L*, *M*, and *N* cyclically follow each other, and δ_n are certain homomorphisms determined by *f* and *g*, called the **connecting homomorphisms**. Topological manifestations of this theorem include

the Mayer–Vietoris sequence and the long exact sequence for relative homology.

Foundational aspects

Cohomology theories have been defined for many different objects such as topological spaces, sheaves, groups, rings, Lie algebras, and C*-algebras. The study of modern algebraic geometry would be almost unthinkable without sheaf cohomology.

Central to homological algebra is the notion of exact sequence; these can be used to perform actual calculations. A classical tool of homological algebra is that of derived functor; the most basic examples are functors Ext and Tor.

With a diverse set of applications in mind, it was natural to try to put the whole subject on a uniform basis. There were several attempts before the subject settled down. An approximate history can be stated as follows:

- → Cartan-→ Eilenberg: In their 1956 book "Homological Algebra", these authors used projective and injective module resolutions.
- 'Tohoku': The approach in a celebrated paper by → Alexander Grothendieck which appeared in the Second Series of the Tohoku Mathematical Journal in 1957, using the abelian category concept (to include sheaves of abelian groups).
- The derived category of Grothendieck and Verdier. Derived categories date back to Verdier's 1967 thesis. They are examples of triangulated categories used in a number of modern theories.

These move from computability to generality.

The computational sledgehammer *par excellence* is the spectral sequence; these are essential in the Cartan-Eilenberg and Tohoku approaches where they are needed, for instance, to compute the derived functors of a composition of two functors. Spectral sequences are less essential in the derived category approach, but still play a role whenever concrete computations are necessary.

There have been attempts at 'non-commutative' theories which extend first cohomology as *torsors* (important in Galois cohomology).

See also

- Homotopical algebra

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Cohomology theory

In mathematics, specifically in → algebraic topology, **cohomology** is a general term for a sequence of abelian groups defined from a cochain complex. That is, cohomology is defined as the abstract study of **cochains**, cocycles, and coboundaries. Cohomology can be viewed as a method of assigning algebraic invariants to a topological space that has a more refined algebraic structure than does homology. Cohomology arises from the algebraic dualization of the construction of homology. In less abstract language, cochains in the fundamental sense should assign 'quantities' to the *chains* of homology theory.

From its beginning in → topology, this idea became a dominant method in the mathematics of the second half of the twentieth century; from the initial idea of *homology* as a topologically invariant relation on *chains*, the range of applications of homology and cohomology theories has spread out over geometry and abstract algebra. The terminology tends to mask the fact that in many applications *cohomology*, a contravariant theory, is more natural than *homology*. At a basic level this has to do with functions and pullbacks in geometric situations: given spaces X and Y, and some kind of function F on Y, for any mapping $f\colon X \to Y$ composition with f gives rise to a function $F \circ f$ on X. Cohomology groups often also have a natural product, the cup product, which gives them a ring structure.

Definition

For a topological space X, the **cohomology group** $H^n(X;G)$, with coefficents in G, is defined to be the quotient Ker(δ)/Im(δ) at $C^n(X;G)$ in the cochain complex

$$\ldots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta} C^n(X;G) \leftarrow \ldots \leftarrow C^0(X;G) \leftarrow 0.$$

Elements in Ker(δ) are **cocycles** and elements in Im(δ) are **coboundaries**.

History

Although cohomology is fundamental to modern → algebraic topology, its importance was not seen for some 40 years after the development of homology. The concept of *dual cell structure*, which Henri Poincaré used in his proof of his → Poincaré duality theorem, contained the germ of the idea of cohomology, but this was not seen until later.

There were various precursors to cohomology. In the mid-1920s, J.W. Alexander and Solomon Lefschetz founded the intersection theory of cycles on manifolds. On an n-dimensional manifold M, a p-cycle and a q-cycle with nonempty intersection will, if in general position, have intersection a $(p+q-n)$-cycle. This enables us to define a multiplication of homology classes

$$H_p(M) \times H_q(M) \to H_{p+q-n}(M).$$

Alexander had by 1930 defined a first cochain notion, based on a p-cochain on a space X having relevance to the small neighborhoods of the diagonal in X^{p+1}.

In 1931, Georges de Rham related homology and exterior differential forms, proving De Rham's theorem. This result is now understood to be more naturally interpreted in terms of cohomology.

In 1934, Lev Pontryagin proved the Pontryagin duality theorem; a result on topological groups. This (in rather special cases) provided an interpretation of → Poincaré duality and Alexander duality in terms of group characters.

At a 1935 conference in Moscow, Andrey Kolmogorov and Alexander both introduced cohomology and tried to construct a cohomology product structure.

In 1936 Norman Steenrod published a paper constructing Čech cohomology by dualizing Čech homology.

From 1936 to 1938, → Hassler Whitney and Eduard Čech developed the cup product (making cohomology into a graded ring) and cap product, and realized that Poincaré duality can be stated in terms of the cap product. Their theory was still limited to finite cell complexes.

In 1944, → Samuel Eilenberg overcame the technical limitations, and gave the modern definition of singular homology and cohomology.

In 1945, Eilenberg and Steenrod stated the axioms defining a homology or cohomology theory. In their 1952 book, *Foundations of Algebraic Topology*, they proved that the existing homology and cohomology theories did indeed satisfy their axioms.[1]

In 1948 Edwin Spanier, building on work of Alexander and Kolmogorov, developed Alexander-Spanier cohomology.

Cohomology theories

Eilenberg-Steenrod theories

A *cohomology theory* is a family of contravariant functors from the category of pairs of topological spaces and continuous functions (or some subcategory thereof such as the category of CW complexes) to the category of Abelian groups and group homomorphisms that satisfies the Eilenberg-Steenrod axioms.

Some cohomology theories in this sense are:

- simplicial cohomology
- singular cohomology
- de Rham cohomology
- Čech cohomology
- sheaf cohomology.

Extraordinary cohomology theories

When one axiom (*dimension axiom*) is relaxed, one obtains the idea of **extraordinary cohomology theory** or **generalized cohomology theory**; this allows theories based on → K-theory and cobordism theory. There are others, coming from stable homotopy theory. In this context, singular homology is referred to as **ordinary homology.**

An extraordinary cohomology theory is "determined by its values on a point", in the sense that if one has a space given by contractible spaces (homotopy equivalent to a point), glued together along contractible spaces, as in a simplicial complex, then the cohomology of the space is determined by the cohomology of a point and the combinatorics of the patching, and effectively computable. Formally, this is computed by the excision theorem, or equivalently the Mayer–Vietoris sequence. Thus the cohomology of a point is a fundamental calculation for any extraordinary cohomology theory, though the cohomology of particular spaces is also of interest.

Other cohomology theories

Theories in a broader sense of *cohomology* include:[2]

- André–Quillen cohomology
- BRST cohomology
- Bonar-Claven cohomology
- Bounded cohomology
- Coherent cohomology
- Crystalline cohomology
- Cyclic cohomology
- Deligne cohomology
- Dirac cohomology
- Flat cohomology
- Galois cohomology
- Gel'fand-Fuks cohomology
- Group cohomology
- Harrison cohomology
- Hochschild cohomology
- Intersection cohomology
- Lie algebra cohomology
- Local cohomology
- Motivic cohomology
- Non-abelian cohomology
- Perverse cohomology
- Quantum cohomology
- Schur cohomology
- Spencer cohomology
- Topological André-Quillen cohomology
- Topological Cyclic cohomology
- Topological Hochschild cohomology
- Γ cohomology
- Étale cohomology

See also

- List of cohomology theories

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K-theory

In mathematics, **K-theory** is a tool used in several disciplines. In → algebraic topology, it is an extraordinary cohomology theory known as topological K-theory. In algebra and algebraic geometry, it is referred to as → algebraic K-theory. It also has some applications in operator algebras. It leads to the construction of families of *K*-functors, which contain useful but often hard-to-compute information.

In physics, K-theory and in particular twisted K-theory have appeared in Type II string theory where it has been conjectured that they classify D-branes, Ramond-Ramond field strengths and also certain spinors on generalized complex manifolds. For details, see also K-theory (physics).

Early history

The subject was originally discovered by → Alexander Grothendieck (1957) so that he could formulate his Grothendieck-Riemann-Roch theorem. It takes its name from the German "Klasse", meaning "class".[1] Grothendieck needed to work with sheaves on an algebraic variety *X*. Rather than working directly with the sheaves, he gave two constructions. In the first, he used the operation of direct sum to convert the commutative monoid of sheaves into a group $K(X)$ by taking formal sums of classes of sheaves and formally adding inverses. (This is an explicit way of obtaining a left adjoint to a certain functor.) In the second construction, he imposed additional relations corresponding to extensions of sheaves to obtain a group now written as $G(X)$. Either of these two constructions is referred to as the Grothendieck group; $K(X)$ has cohomological behavior and $G(X)$ has homological behavior.

If X is a smooth variety, the two groups are the same.

In topology, one has an analogous sum construction for vector bundles. → Michael Atiyah and Friedrich Hirzebruch used the Grothendieck group construction to define $K(X)$ for a topological space X in 1959. This was the basis of the first extraordinary cohomology theory discovered in → algebraic topology. It played a major role in the second proof of the Index Theorem (circa 1962). Furthermore this approach led to a noncommutative K-theory for C*-algebras.

Already in 1955, → Jean-Pierre Serre had used the analogy of vector bundles with projective modules to formulate Serre's conjecture, which states that projective modules over the ring of polynomials over a field are free modules; this assertion is correct, but was not settled until 20 years later. (Swan's theorem is another aspect of this analogy.) In 1959, Serre formed the Grothendieck group construction for rings, and used it to show that projective modules are stably free. This application was the beginning of → **algebraic K-theory**.

Developments

There followed a period in which there were various partial definitions of *higher K-theory functors*. Finally, two useful and equivalent definitions were given by → Daniel Quillen using → homotopy theory in 1969 and 1972. A variant was also given by Friedhelm Waldhausen in order to study the *algebraic K-theory of spaces,* which is related to the study of pseudo-isotopies. Most modern research on higher K-theory is related to algebraic geometry and the study of motivic cohomology.

The corresponding constructions involving an auxiliary quadratic form receive the general name L-theory. It is a major tool of surgery theory.

In string theory the K-theory classification of Ramond-Ramond field strengths and the charges of stable D-branes was first proposed in 1997.[2]

See also

- List of cohomology theories
- K-theory (physics)
- L-theory
- Bott periodicity

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Algebraic K-theory

In mathematics, **algebraic K-theory** is an advanced part of → homological algebra concerned with defining and applying a sequence

$$K_n(R)$$

of functors from rings to abelian groups, for all integers *n*. For historical reasons, the **lower K-groups** K_0 and K_1 are thought of in somewhat different terms from the **higher algebraic K-groups** K_n for $n \geq 2$. Indeed, the lower groups are more accessible, and have more applications, than the higher groups. The theory of the higher K-groups is noticeably deeper, and certainly much harder to compute (even when R is the ring of integers).

The group $K_0(R)$ generalises the construction of the ideal class group of a ring, using projective modules. Its development in the 1960s and 1970s was linked to attempts to solve a conjecture of → Serre on projective modules that now is the Quillen-Suslin theorem; numerous other connections with classical algebraic problems were found in this era. Similarly, $K_1(R)$ is a modification of the group of units in a ring, using elementary matrix theory. The group $K_1(R)$ is important in → topology, especially when R is a group ring, because its quotient the Whitehead group contains the Whitehead torsion used to study problems in simple homotopy theory and surgery theory; the group $K_0(R)$ also contains other invariants such as the finiteness invariant. Since the 1980s, algebraic K-theory has increasingly had applications to algebraic geometry. For example, motivic cohomology is closely related to algebraic *K*-theory.

History

Alexander Grothendieck invented K-theory in the mid-1950s as a framework to state his far-reaching generalization of the Riemann-Roch theorem. Within a few years, its topological counterpart was considered by Atiyah and Hirzebruch and is now known as topological K-theory.

Applications of *K*-groups were found from 1960 onwards in surgery theory for manifolds, in particular; and numerous other connections with classical algebraic problems were brought out.

A little later a branch of the theory for operator algebras was fruitfully developed, resulting in operator K-theory and KK-theory. It also became clear that *K*-theory could play a role in algebraic cycle theory in algebraic geometry (Gersten's conjecture): here the *higher K*-groups become connected with the *higher codimension* phenomena, which are exactly those that are harder to access. The problem was that the definitions were lacking (or, too many and not obviously consistent). A definition of K_2 for fields by John Milnor, for example, gave an attractive theory that was too limited in scope, constructed as a quotient of the multiplicative group of the field tensored with itself, with some explicit relations imposed; and closely connected with central extensions.

Eventually the foundational difficulties were resolved (leaving a deep and difficult theory) by → Daniel Quillen, who gave several definitions of higher algebraic *K*-theory, via the +-construction and the *Q*-construction.

Lower K-groups

The lower K-groups were discovered first, and given various ad hoc descriptions, which remain useful. Throughout, let *A* be a ring.

K_0

The 0th K-group is related to dimension and the Picard group.

The (covariant) functor K_0 goes from the category of rings to the category of groups, taking *A* to the Grothendieck group of the set of isomorphism classes of its finitely generated projective modules, regarded as a monoid under direct sum.

(Projective) modules over a field *k* are vector spaces and K_0(k) is isomorphic to $\mathbf{Z}$, by dimension. For *A* a Dedekind ring,

$$K_0(A) = \mathrm{Pic}A \times \mathbf{Z},$$

where Pic(*A*) is the Picard group of *A*, and similarly the reduced K-theory is given by

$$\tilde{K}_0(A) = \mathrm{Pic}A.$$

K_1

Hyman Bass provided this definition, which generalizes the group of units of a field: $K_1(A)$ is the abelianization of the infinite general linear group:

$$K_1(A) = \mathrm{GL}(A)^{\mathrm{ab}} = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)]$$

Here

$$\mathrm{GL}(A) = \mathrm{colim}\mathrm{GL}_n(A)$$

is the direct limit of the GL_n, which embeds in GL_{n+1} as the upper left block matrix, and the commutator subgroup agrees with the group generated by elementary matrices $\mathrm{E}(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$, by Whitehead's lemma. Indeed, the group $\mathrm{GL}(A)/\mathrm{E}(A)$ was first defined and studied by Whitehead,[1] and is called the **Whitehead group** of the *ring*[2] *A*.

As $\mathrm{E}(A) \triangleleft \mathrm{SL}(A)$, one can also define the **special Whitehead group** $SK_1(A) := \mathrm{SL}(A)/\mathrm{E}(A)$.

Commutative rings and fields

For *A* a commutative ring, one can define a determinant $\det\colon \mathrm{GL}(A) \to A^*$ to the group of units of *A*, which vanishes on $\mathrm{E}(A)$ and thus descends to a map $\det\colon K_1(A) \to A^*$. This map splits via the map $A^* \xrightarrow{\sim} GL_1(A) \to K_1(A)$ (unit in the upper left corner), and hence is onto, and has the special Whitehead group as kernel, yielding the split short exact sequence:

$$1 \to SK_1(A) \to K_1(A) \to A^* \to 1,$$

which is a quotient of the usual split short exact sequence defining the special linear group, namely

$$1 \to \mathrm{SL}(A) \to \mathrm{GL}(A) \to A^* \to 1.$$

Thus, since the groups in question are abelian, $K_1(A)$ splits as the direct sum of the group of units and the special Whitehead group: $K_1(A) \approx A^* \oplus SK_1(A)$.

When *A* is a Euclidean domain (e.g. a field, or the integers) $SK_1(A)$ vanishes, and the determinant map is an isomorphism. In particular, $\det\colon K_1(F) \xrightarrow{\sim} F^*$. This is *false* in general for PIDs, thus providing one of the rare mathematical features of Euclidean domains that do not generalize to all PIDs. An explicit PID A such that $SK_1(A)$ is nonzero was given by Grayson in 1981. A hard theorem of Bass, Milnor, and Serre shows $SK_1(A)$ vanishes when *A* is the ring of *S*-integers in any global field.

For a non-commutative ring, the determinant cannot be defined, but the map $\mathrm{GL}(A) \to K_1(A)$ generalizes the determinant.

K$_2$

John Milnor found the right definition of K_2: it is the center of the Steinberg group $\mathrm{St}(A)$ of A.
It can also be defined as the kernel of the map

$$\varphi\colon \mathrm{St}(A) \to \mathrm{GL}(A),$$

or as the Schur multiplier of the group of elementary matrices.

For a field k one has

$$K_2(k) = k^\times \otimes_{\mathbf{Z}} k^\times / \langle a \otimes (1-a) \mid a \neq 0, 1 \rangle.$$

Milnor *K*-theory

The above expression for K_2 of a field k led Milnor to the following definition of "higher" K-groups by

$$K_*^M(k) := T^* k^\times / (a \otimes (1-a)),$$

thus as graded parts of a quotient of the tensor algebra of the multiplicative group $k^\times$ by the two-sided ideal, generated by the

$$a \otimes (1-a)$$

for $a \neq 0,1$. For $n = 0,1,2$ these coincide with those above, but for $n \geq 3$ they differ in general. For example, we have $K_n^M(\mathbb{F}_q) = 0$ for $n \geq 3$.

Higher *K*-theory

The master, definitive definitions of K-theory were given by → Daniel Quillen, after an extended period in which uncertainty had reigned.

The +-construction

One possible definition of higher algebraic K-theory of rings was given by Quillen

$$K_n(R) = \pi_n(BGL(R)^+),$$

a very compressed piece of abstract mathematics. Here π_k is a homotopy group, $GL(R)$ is the direct limit of the general linear groups over R for the size of the matrix tending to infinity, B is the classifying space construction of → homotopy theory, and the $^+$ is Quillen's plus construction.

This definition only holds for $n>0$ so one often defines the higher algebraic K-theory via

$$K_n(R) = \pi_n(BGL(R)^+ \times K_0(R))$$

Since $BGL(R)^+$ is path connected and $K_0(R)$ discrete, this definition doesn't differ in higher degrees and also holds for $n=0$.

The Q-construction

The Q-construction gives the same results as the +-construction, but it applies in more general situations. Moreover, the definition is more direct in the sense that the *K*-groups, defined via the Q-construction are functorial by definition. This fact is not automatic in the +-construction.

Suppose *P* is an exact category; associated to *P* a new category Q*P* is defined, objects of which are those of *P* and morphisms from *M′* to *M″* are isomorphism classes of exact diagrams

$$M' \longleftarrow N \longrightarrow M''.$$

where the first arrow is an admissible epimorphism and the second arrow is an admissible monomorphism.

The *i*-th ***K*-group** of *P* is then defined as

$$K_i(P) = \pi_{i+1}(\mathrm{B}\mathrm{Q}P, 0)$$

with a fixed zero-object 0, where B*Q* is the *classifying space* of *Q*, which is defined to be the geometric realisation of the *nerve* of *Q*.

This definition coincides with the above definitions of K_0, K_1 and K_2.

The *K*-groups $K_i(A)$ of the ring *A* are then the *K*-groups $K_i(P_A)$ where P_A is the category of finitely generated projective *A*-modules. More generally, for a scheme *X*, the higher *K*-groups of *X* are by definition the *K*-groups of (the exact category of) locally free coherent sheaves on *X*.
The following variant of this is also used: instead of finitely generated projective (=locally free) modules, take finitely generated modules. The resulting *K*-groups are usually called *G*-groups, or *higher G-theory*. When *A* is a noetherian regular ring, then *G*- and *K*-theory coincide. Indeed, the global dimension of regular local rings is finite, i.e. any finitely generated module has a finite projective resolution, so the canonical map $K_0 \to G_0$ is surjective. It is also injective, as can be shown. This isomorphism extends to the higher *K*-groups, too.

Examples

While the Quillen algebraic K-theory has provided deep insight into various aspects of algebraic geometry and topology, the K-groups have proved particularly difficult to compute except in a few isolated but interesting cases.

Algebraic K-groups of finite fields

The first and one of the most important calculations of the higher algebraic K-groups of a ring were made by Quillen himself for the case of finite fields:

Theorem. Let *F* be a finite field with *q* elements. Then

$$K_0(F) = Z\,,\ K_{2i}(F) = 0$$

for $i \neq 0$, and

$$K_{2i-1}(F) = \mu_{q^i-1} \text{ for } i = 1, 2, \ldots$$

where μ_r denotes the cyclic group with *r* elements.

Algebraic K-groups of rings of integers

Quillen proved that if *A* is the ring of algebraic integers in an algebraic number field *F* (a finite extension of the rationals), then the algebraic K-groups of *A* are finitely generated. Borel used this to calculate $K_i(A)$ and $K_i(F)$ modulo torsion. For example, for the integers **Z**, Borel proved that (modulo torsion)

for positive *i* unless $i = 4k + 1$ with *k* positive

and (modulo torsion)

for positive *k*.

The torsion subgroups of $K_{2i+1}(\mathbf{Z})$, and the orders of the finite groups $K_{4k+2}(\mathbf{Z})$ have recently been determined, but whether the latter groups are cyclic, and whether the groups $K_{4k}(\mathbf{Z})$ vanish depends upon Vandiver's conjecture about the class groups of cyclotomic integers.

Applications

Algebraic K-groups are used in conjectures on special values of L-functions and the formulation of an non-commutative main conjecture of Iwasawa theory and in construction of higher regulators.

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External links

- C. Weibel " The K-book: An introduction to algebraic K-theory (http://www.math.rutgers.edu/~weibel/Kbook.html)"

TQFT

A **topological quantum field theory** (or **topological field theory** or **TQFT**) is a quantum field theory which computes topological invariants.

Although TQFTs were invented by physicists, they are also of mathematical interest, being related to, among other things, knot theory and the theory of four-manifolds in → algebraic topology, and to the theory of moduli spaces in algebraic geometry. Donaldson, Jones, Witten, and → Kontsevich have all won Fields Medals for work related to topological field theory.

In condensed matter physics, topological quantum field theories are the low energy effective theories of topologically ordered states, such as fractional quantum Hall states, string-net condensed states, and other strongly correlated quantum liquid states.

Overview

In a topological field theory, the correlation functions do not depend on the metric on spacetime. This means that the theory is not sensitive to changes in the shape of spacetime; if the spacetime warps or contracts, the correlation functions do not change. Consequently, they are topological invariants.

Topological field theories are not very interesting on the flat Minkowski spacetime used in particle physics. Minkowski space can be contracted to a point, so a TQFT on Minkowski space computes only trivial topological invariants. Consequently, TQFTs are usually studied on curved spacetimes, such as, for example, Riemann surfaces. Most of the known topological field theories are defined on spacetimes of dimension less than five. It seems that a few higher dimensional theories exist, but they are not very well understood.

Quantum gravity is believed to be background-independent (in some suitable sense), and TQFTs provide examples of background independent quantum field theories. This has prompted ongoing theoretical investigation of this class of models.

(Caveat: It is often said that TQFTs have only finitely many degrees of freedom. This is not a fundamental property. It happens to be true in most of the examples that physicists and mathematicians study, but it is not necessary. A topological sigma model with target infinite-dimensional projective space, if such a thing could be defined, would have countably infinitely many degrees of freedom.)

Specific models

The known topological field theories fall into two general classes: Schwarz-type TQFTs and Witten-type TQFTs. Witten TQFTs are also sometimes referred to as cohomological field theories.

Schwarz-type TQFTs

In Schwarz-type TQFTs, the correlation functions computed by the path integral are topological invariants because the path integral measure and the quantum field observables are explicitly independent of the metric. For instance, in the BF model, the spacetime is a two-dimensional manifold M, the observables are constructed from a two-form F, an auxiliary scalar B, and their derivatives. The action (which determines the path integral) is

$$S = \int_M BF$$

The spacetime metric does not appear anywhere in this theory, so the theory is explicitly topologically invariant. Another, more famous example is Chern-Simons theory, which can be used to compute knot invariants.

Witten-type TQFTs

In Witten-type topological field theories, the topological invariance is more subtle. For example the Lagrangian for the WZW model does depend explicitly on the metric, but one shows by calculation that the expectation value of the partition function and a special class of correlation functions are in fact diffeomorphism invariant.

Mathematical formulations

Atiyah-Segal axioms

→ Atiyah suggested a set of axioms for topological quantum field theory which was inspired by Segal's proposed axioms for conformal field theory, (Atiyah 1988). These axioms have been relatively useful for mathematical treatments of Schwarz-type QFTs, although it isn't clear that they capture the whole structure of Witten-type QFTs. The basic idea is that a TQFT is a functor from a certain category of cobordisms to the category of vector spaces.

There are in fact two different sets of axioms which could reasonably be called the Atiyah axioms. These axioms differ basically in whether or not they study a TQFT defined on a single fixed *n*-dimensional Riemannian / Lorentzian spacetime *M* or a TQFT defined on all *n*-dimensional spacetimes at once.

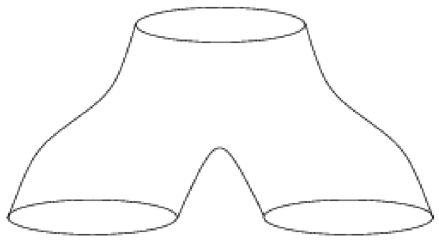
[ed. *What follows is still in rough draft form and should be regarded suspiciously.*]

The case of a fixed spacetime

Let $Bord_M$ be the category whose morphisms are *n*-dimensional submanifolds of *M* and whose objects are connected components of the boundaries of such submanifolds. Regard two morphisms as equivalent if they are homotopic via submanifolds of M, and so form the quotient category $hBord_M$: The objects in $hBord_M$ are the objects of $Bord_M$, and the morphisms of $hBord_M$ are homotopy equivalence classes of morphisms in $Bord_M$. A TQFT on M is a symmetric monoidal functor from $hBord_M$ to the category of vector spaces.
Note that cobordisms can, if their boundaries match up, be sewn together to form a new bordism. This is the composition law for morphisms in the cobordism category. Since functors are required to preserve composition, this says that the linear map corresponding to a sewn together morphism is just the composition of the linear map for each piece.

There is an equivalence of categories between the category of 2-dimensional topological quantum field theories and the category of commutative Frobenius algebras.

All n-dimensional spacetimes at once



The pair of pants is a (1+1)-dimensional bordism, which corresponds to a product or coproduct in a 2-dimensional TQFT.

To consider all spacetimes at once, it is necessary to replace $hBord_M$ by a larger category. So let $Bord_n$ be the category of bordisms, i.e. the category whose morphisms are *n*-dimensional manifolds with boundary, and whose objects are the connected components of the boundaries of n-dimensional manifolds. (Note that any $(n-1)$-dimensional manifold may appear as an object in $Bord_n$.) As above, regard two morphisms in $Bord_n$ as equivalent if they are homotopic, and form the quotient category $hBord_n$. $Bord_n$ is a monoidal category under the operation which takes two bordisms to the bordism made from their disjoint union. A TQFT on *n*-dimensional manifolds is then a functor from $hBord_n$ to the category of vector spaces, which takes disjoint unions of bordisms to the tensor product f [ed. *unfinished*]

For example, for (1+1)-dimensional bordisms (2-dimensional bordisms between 1-dimensional manifolds), the map associated with a pair of pants gives a product or coproduct, depending on how the boundary components are

grouped – which is commutative or cocommutative, while the map associated with a disk gives a counit (trace) or unit (scalars), depending on grouping of boundary, and thus (1+1)-dimension TQFTs correspond to Frobenius algebras.

Generalizations

For some applications, it is convenient to demand extra topological structure on the morphisms, such as a choice of orientation.

See also

- Quantum topology
- Topological defect
- Topological entropy in physics
- Topological order
- Topological quantum number
- Topological string theory
- Arithmetic topology

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[5] http://projecteuclid.org/euclid.cmp/1104161738
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Important publications in algebraic topology

This is a list of **important publications** in mathematics, organized by field.

Some reasons why a particular publication might be regarded as important:

- **Topic creator** – A publication that created a new topic
- **Breakthrough** – A publication that changed scientific knowledge significantly
- **Introduction** – A publication that is a good introduction or survey of a topic
- **Influence** – A publication which has significantly influenced the world
- **Latest and greatest** – The current most advanced result in a topic

Algebra

Theory of equations

Hisab al-Jabr w'al-muqabala, Kitab al-Jabr wa-l-Muqabala

- Muhammad ibn Mūsā al-Khwārizmī (820)

Description: The first book on the systematic algebraic solutions of linear and quadratic equations. The book is considered to be the foundation of modern algebra and Islamic mathematics. The word "algebra" itself is derived from the *al-Jabr* in the title of the book.

Ars Magna

- Gerolamo Cardano (1545)

Description: Provided the first published methods for solving cubic and quartic equations (due to Scipione del Ferro, Niccolò Fontana Tartaglia, and Lodovico Ferrari), and exhibited the first published calculations involving non-real complex numbers. [1]

Vollständige Anleitung zur Algebra

- Leonhard Euler (1770)

Description: Also known as Elements of Algebra, Euler's textbook on elementary algebra is one of the first to set out algebra in the modern form we would recognize today. The first volume deals with determinate equations, while the second part deals with Diophantine equations. The last section contains a proof of Fermat's Last Theorem for the case $n = 3$, making some valid assumptions regarding $Q(\sqrt{-3})$ that Euler did not prove.[2]

Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse

- Carl Friedrich Gauss (1799)

Description: Gauss' doctoral dissertation,[3] which contained a widely accepted (at the time) but incomplete proof[4] of the fundamental theorem of algebra.

Abstract algebra

Group theory

Réflexions sur la résolution algébrique des équations

- Joseph Louis Lagrange (1770)

Description: Made the prescient observation that the roots of the Lagrange resolvent of a polynomial equation are tied to permutations of the roots of the original equation, laying a more general foundation for what had previously been an ad hoc analysis and helping motivate the later development of the theory of permutation groups, group theory, and Galois theory. The Lagrange resolvent also introduced the discrete Fourier transform of order 3.

Articles Publiés par Galois dans les Annales de Mathématiques

- Journal de Mathematiques pures et Appliquées, II (1846)

Description: Posthumous publication of the mathematical manuscripts of Évariste Galois by Joseph Liouville. Included are Galois' papers *Mémoire sur les conditions de résolubilité des équations par radicaux* and *Des équations primitives qui sont solubles par radicaux*.

Traité des substitutions et des équations algébriques

- Camille Jordan (1870)

Description: The first book on group theory, giving a then-comprehensive study of permutation groups and Galois theory. In this book, Jordan introduced the notion of a simple group and epimorphism (which he called *l'isomorphisme mériédrique*),[5] proved part of the Jordan–Hölder theorem, and discussed matrix groups over finite fields as well as the Jordan normal form. [6]

Theorie der Transformationsgruppen

- Sophus Lie, Friedrich Engel (1888-1893).

Publication data: 3 volumes, B.G. Teubner, Verlagsgesellschaft, mbH, Leipzig, 1888-1893. Volume 1 [7], Volume 2 [8], Volume 3 [8].

Description: The first comprehensive work on transformation groups, serving as the foundation for the modern theory of Lie groups.

Solvability of groups of odd order

- Walter Feit and John Thompson (1960)

Description: Gave a complete proof of the solvability of finite groups of odd order, establishing the long-standing Burnside conjecture that all finite non-abelian simple groups are of even order. Many of the original techniques used in this paper were used in the eventual classification of finite simple groups.

→ Homological algebra

Homological Algebra

- → Henri Cartan and → Samuel Eilenberg (1956)

Description: Provided the first fully-worked out treatment of abstract homological algebra, unifying previously disparate presentations of homology and cohomology for associative algebras, Lie algebras, and groups into a single theory.

Sur Quelques Points d'Algèbre Homologique

- → Alexander Grothendieck (1957)

Description: Revolutionized → homological algebra by introducing abelian categories and providing a general framework for Cartan and Eilenberg's notion of derived functors.

Algebraic geometry

Theorie der Abelschen Functionen

- Bernhard Riemann (1857)

Publication data: *Journal für die Reine und Angewandte Mathematik*

Description: Developed the concept of Riemann surfaces and their topological properties beyond Riemann's 1851 thesis work, proved an index theorem for the genus (the original formulation of the Riemann-Hurwitz formula), proved the Riemann inequality for the dimension of the space of meromorphic functions with prescribed poles (the original formulation of the Riemann-Roch theorem), discussed birational transformations of a given curve and the dimension of the corresponding moduli space of inequivalent curves of a given genus, and solved more general inversion problems than those investigated by Abel and Jacobi. André Weil once wrote that this paper "*is one of the greatest pieces of mathematics that has ever been written; there is not a single word in it that is not of consequence.*" [7]

Faisceaux Algébriques Cohérents

- → Jean-Pierre Serre

Publication data: *Annals of Mathematics*, 1955

Description: *FAC*, as it is usually called, first introduced the use of sheaves into algebraic geometry. Serre introduced Čech cohomology of sheaves in this paper, and, despite its technical deficiencies, revolutionized algebraic geometry. For example, the long exact sequence in sheaf cohomology allows one to show that some surjective maps of sheaves induce surjective maps on sections; specifically, these are the maps whose kernel (as a sheaf) has a vanishing first cohomology group. Before *FAC*, this was next to impossible. While Grothendieck's derived functor cohomology has replaced Čech cohomology for technical reasons, actual calculations, such as of the cohomology of projective space, are usually carried out by Čech techniques, and for this reason Serre's paper remains important even today.

Géométrie Algébrique et Géométrie Analytique

- → Jean-Pierre Serre (1956)

Description: In mathematics, algebraic geometry and analytic geometry are closely related subjects, where *analytic geometry* is the theory of complex manifolds and the more general analytic spaces defined locally by the vanishing of analytic functions of several complex variables. A (mathematical) theory of the relationship between the two was put in place during the early part of the 1950s, as part of the business of laying the foundations of algebraic geometry to include, for example, techniques from Hodge theory. (*NB* While analytic geometry as use of Cartesian coordinates is also in a sense included in the scope of algebraic geometry, that is not the topic being discussed in this article.) The major paper consolidating the theory was *Géometrie Algébrique et Géométrie Analytique* by → Serre, now usually referred to as *GAGA*. A *GAGA-style result* would now mean any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, and a well-defined subcategory of analytic geometry objects and holomorphic mappings.

Le théorème de Riemann-Roch, d'après A. Grothendieck

- Armand Borel, → Jean-Pierre Serre (1958)

Description: Borel and Serre's exposition of Grothendieck's version of the Riemann Roch theorem, published after Grothendieck made it clear that he was not interested in writing up his own result. Grothendieck reinterpreted both sides of the formula that Hirzebruch proved in 1953 in the framework of morphisms between varieties, resulting in a sweeping generalization.[8] In his proof, Grothendieck broke new ground with his concept of Grothendieck groups, which led to the development of → K-theory.[9]

Éléments de géométrie algébrique

- → Alexander Grothendieck (1960-1967)

Description: Written with the assistance of Jean Dieudonné, this is Grothendieck's exposition of his reworking of the foundations of algebraic geometry. It has become the most important foundational work in modern algebraic geometry. The approach expounded in EGA, as these books are known, transformed the field and led to monumental advances.

Séminaire de géométrie algébrique

- → Alexander Grothendieck et al.

Description: These seminar notes on Grothendieck's reworking of the foundations of algebraic geometry report on work done at IHÉS starting in the 1960s. SGA 1 dates from the seminars of 1960-1961, and the last in the series, SGA 7, dates from 1967–1969. In contrast to EGA, which is intended to set foundations, SGA describes ongoing research as it unfolded in Grothendieck’s seminar; as a result, it is quite difficult to read, since many of the more elementary and foundational results were relegated to EGA. One of the major results building on the results in SGA is Pierre Deligne's proof of the last of the open Weil conjectures in the early 1970s. Other authors who worked on one or several volumes of SGA include Michel Raynaud, Michael Artin, → Jean-Pierre Serre, Jean-Louis Verdier, Pierre Deligne, and Nicholas Katz.

Number theory

De fractionibus continuis dissertatio

- Leonhard Euler (1744)

Description: First presented in 1737, this paper [10] provided the first then-comprehensive account of the properties of continued fractions. It also contains the first proof that the number e is irrational.[11]

Recherches d'Arithmétique

- Joseph Louis Lagrange (1775)

Description: Developed a general theory of binary quadratic forms to handle the general problem of when an integer is representable by the form $ax^2 + by^2 + cxy$. This included a reduction theory for binary quadratic forms, where he proved that every form is equivalent to a certain canonically chosen reduced form.[12] [13]

Disquisitiones Arithmeticae

- Carl Friedrich Gauss (1801)

Description: The *Disquisitiones Arithmeticae* is a profound and masterful book on number theory written by German mathematician Carl Friedrich Gauss and first published in 1801 when Gauss was 24. In this book Gauss brings together results in number theory obtained by mathematicians such as Fermat, Euler, Lagrange and Legendre and adds many important new results of his own. Among his contributions was the first complete proof known of the Fundamental theorem of arithmetic, the first two published proofs of the law of quadratic reciprocity, a deep investigation of binary quadratic forms going beyond Lagrange's work in Recherches d'Arithmétique, a first appearance of Gauss sums, cyclotomy, and the theory of constructible polygons with a particular application to the constructibility of the regular 17-gon. Of note, in section V, article 303 of Disquisitiones, Gauss summarized his calculations of class numbers of imaginary quadratic number fields, and in fact found all imaginary quadratic number fields of class numbers 1, 2, and 3 (confirmed in 1986) as he had conjectured.[14] In section V, article 358, Gauss proved what can be interpreted as the first non-trivial case of the Riemann Hypothesis for curves over finite fields (the Hasse-Weil theorem).[15]

Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthä lt

- Johann Peter Gustav Lejeune Dirichlet (1837)

Description: Pioneering paper in analytic number theory, which introduced Dirichlet characters and their L-functions to establish Dirichlet's theorem on arithmetic progressions.[16] In subsequent publications, Dirichlet used these tools to determine, among other things, the class number for quadratic forms.

Über die Anzahl der Primzahlen unter einer gegebenen Grösse

- Bernhard Riemann (1859)

Description: *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (or *On the Number of Primes Less Than a Given Magnitude*) is a seminal 8-page paper by Bernhard Riemann published in the November 1859 edition of the *Monthly Reports of the Berlin Academy*. Although it is the only paper he ever published on number theory, it contains ideas which influenced dozens of researchers during the late 19th century and up to the present day. The paper consists primarily of definitions, heuristic arguments, sketches of proofs, and the application of powerful analytic methods; all of these have become essential concepts and tools of modern analytic number theory. It also contains the famous Riemann Hypothesis, one of the most important open problems in mathematics.

Vorlesungen über Zahlentheorie

- P.G.L. Dirichlet and Richard Dedekind

Description: *Vorlesungen über Zahlentheorie* (*Lectures on Number Theory*) is a textbook of number theory written by German mathematicians P.G.L. Dirichlet and Richard Dedekind, and published in 1863. The *Vorlesungen* can be seen as a watershed between the classical number theory of Fermat, Jacobi and Gauss, and the modern number theory of Dedekind, Riemann and Hilbert. Dirichlet does not explicitly recognise the concept of the group that is central to modern algebra, but many of his proofs show an implicit understanding of group theory

Zahlbericht

- David Hilbert (1897)

Description: Unified and made accessible many of the developments in algebraic number theory made during the nineteenth century. Although criticized by André Weil (who stated "*more than half of his famous Zahlbericht is little more than an account of Kummer's number-theoretical work, with inessential improvements*")[17] and Emmy Noether,[18] it was highly influential for many years following its publication.

Fourier Analysis in Number Fields and Hecke's Zeta-Functions

- John Tate (1950)

Description: Generally referred to simply as *Tate's Thesis*, Tate's Princeton Ph.D. thesis, under Emil Artin, is a reworking of Erich Hecke's theory of zeta- and *L*-functions in terms of Fourier analysis on the adeles. The introduction of these methods into number theory made it possible to formulate extensions of Hecke's results to more general *L*-functions such as those arising from automorphic forms.

Automorphic Forms on GL(2)

- Hervé Jacquet and Robert Langlands (1970)

Description: This publication offers evidence towards Langlands' conjectures by reworking and expanding the classical theory of modular forms and their *L*-functions through the introduction of representation theory.

La conjecture de Weil. I.

- Pierre Deligne (1974)

Description: Proved the Riemann hypothesis for varieties over finite fields, settling the last of the open Weil conjectures.

Endlichkeitssätze für abelsche Varietäten über Zahlkörpern

- Gerd Faltings (1983)

Description: Faltings proves a collection of important results in this paper, the most famous of which is the first proof of the Mordell conjecture (a conjecture dating back to 1922). Other theorems proved in this paper include an instance of the Tate conjecture (relating the homomorphisms between two abelian varieties over a number field to the homomorphisms between their Tate modules) and some finiteness results concerning abelian varieties over number fields with certain properties.

Modular Elliptic Curves and Fermat's Last Theorem

- Andrew Wiles (1995)

Description: This article proceeds to prove a special case of the Shimura-Taniyama conjecture through the study of the deformation theory of Galois representations. This in turn implies the famed Fermat's Last Theorem. The proof's method of identification of a deformation ring with a Hecke algebra (now referred to as an $R=T$ theorem) to prove modularity lifting theorems has been an influential development in algebraic number theory.

The geometry and cohomology of some simple Shimura varieties

- Michael Harris and Richard Taylor (2001)

Description: Harris and Taylor provide the first proof of the local Langlands conjecture for GL(n). As part of the proof, this monograph also makes an in depth study of the geometry and cohomology of certain Shimura varieties at primes of bad reduction.

Analysis

Introductio in analysin infinitorum

- Leonhard Euler (1748)

Description: The eminent historian of mathematics Carl Boyer once called Euler's *Introductio in analysin infinitorum* the greatest modern textbook in mathematics.[19] Published in two volumes,[20] [21] this book more than any other work succeeded in establishing analysis as a major branch of mathematics, with a focus and approach distinct from that used in geometry and algebra.[22] Notably, Euler identified functions rather than curves to be the central focus in his book.[23] Logarithmic, exponential, trigonometric, and transcendental functions were covered, as were expansions into partial fractions, evaluations of $\zeta(2k)$ for k a positive integer between 1 and 13, infinite series-infinite product formulas,[19] continued fractions, and partitions of integers.[24] In this work, Euler proved that every rational number can be written as a finite continued fraction, that the continued fraction of an irrational number is infinite, and derived continued fraction expansions for e and $\sqrt{e}$.[20] This work also contains a statement of Euler's formula and a statement of the pentagonal number theorem, which he had discovered earlier and would publish a proof for in 1751.

Calculus

Yuktibhasa

- Jyeshtadeva (1501)

Description: Written in India in 1501, this was the world's first calculus text. "This work laid the foundation for a complete system of fluxions" (Charles Whish, 1835) and served as a summary of the Kerala School's achievements in calculus, trigonometry and mathematical analysis, most of which were earlier discovered by the 14th century mathematician Madhava. It's possible that this text influenced the later development of calculus in Europe. Some of its important developments in calculus include: the fundamental ideas of differentiation and integration, the derivative, differential equations, term by term integration, numerical integration by means of infinite series, the relationship between the area of a curve and its integral, and the mean value theorem.

Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus

- Gottfried Leibniz (1684)

Description: Leibniz's first publication on differential calculus, containing the now familiar notation for differentials as well as rules for computing the derivatives of powers, products and quotients.

Philosophiae Naturalis Principia Mathematica

- Isaac Newton

Description: The ***Philosophiae Naturalis Principia Mathematica*** (Latin: "mathematical principles of natural philosophy", often *Principia* or *Principia Mathematica* for short) is a three-volume work by Isaac Newton published on July 5, 1687. Perhaps the most influential scientific book ever published, it contains the statement of Newton's laws of motion forming the foundation of classical mechanics as well as his law of universal gravitation, and derives Kepler's laws for the motion of the planets (which were first obtained empirically). Here was born the practice, now so standard we identify it with science, of explaining nature by postulating mathematical axioms and demonstrating that their conclusion are observable phenomena. In formulating his physical theories, Newton freely used his unpublished work on calculus. When he submitted Principia for publication, however, Newton chose to recast the majority of his proofs as geometric arguments.[25]

Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum

- Leonhard Euler (1755)

Description: Published in two books,[26] Euler's textbook on differential calculus presented the subject in terms of the function concept, which he had introduced in his 1748 *Introductio in analysin infinitorum*. This work opens with a study of the calculus of finite differences and makes a thorough investigation of how differentiation behaves under substitutions.[1] Also included is a systematic study of Bernoulli polynomials and the Bernoulli numbers (naming them as such), a demonstration of how the Bernoulli numbers are related to the coefficients in the Euler–Maclaurin formula and the values of $\zeta(2n)$,[27] a further study of Euler's constant (including its connection to the gamma function), and an application of partial fractions to differentiation. [28]

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

- Bernhard Riemann (1867)

Description: Written in 1853, Riemann's work on trigonometric series was published posthumously. In it, he extended Cauchy's definition of the integral to that of the Riemann integral, allowing some functions with dense subsets of discontinuities on an interval to be integrated (which he demonstrated by an example).[29] He also stated the Riemann series theorem,[29] proved the Riemann-Lebesgue lemma for the case of bounded Riemann integrable functions,[30] and developed the Riemann localization principle.[31]

Intégrale, longueur, aire

- Henri Lebesgue (1901)

Description: Lebesgue's doctoral dissertation, summarizing and extending his research to date regarding his development of measure theory and the Lebesgue integral.

Complex analysis

Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse

- Bernhard Riemann (1851)

Description: Riemann's doctoral dissertation introduced the notion of a Riemann surface, conformal mapping, simple connectivity, the Riemann sphere, the Laurent series expansion for functions having poles and branch points, and the Riemann mapping theorem.

Functional analysis

Théorie des opérations linéaires

- Stefan Banach (1932; originally published 1931 in Polish under the title *Teorja operacyj*.)

Description: The first mathematical monograph on the subject of linear metric spaces, bringing the abstract study of functional analysis to the wider mathematical community. The book introduced the ideas of a normed space and the notion of a so-called *B*-space, a complete normed space. The *B*-spaces are now called Banach spaces and are one of the basic objects of study in all areas of modern mathematical analysis. Banach also gave proofs of versions of the open mapping theorem, closed graph theorem, and Hahn-Banach theorem.

Fourier analysis

Mémoire sur la propagation de la chaleur dans les corps solides

- Joseph Fourier (1807)[32]

Description: Introduced Fourier analysis, specifically Fourier series. Key contribution was to not simply use trigonometric series, but to model *all* functions by trigonometric series.

> “$\varphi(y) = a\cos\frac{\pi y}{2} + a'\cos 3\frac{\pi y}{2} + a''\cos 5\frac{\pi y}{2} + \cdots.$
> Multiplying both sides by $\cos(2i+1)\frac{\pi y}{2}$, and then integrating from $y=-1$ to $y=+1$ yields:
> $a_i = \int_{-1}^{1} \varphi(y)\cos(2i+1)\frac{\pi y}{2}\,dy.$”

When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: ...*the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour*. Making Fourier series rigorous, which in detail took over a century, led directly to a number of developments in analysis, notably the rigorous statement of the integral via the Dirichlet integral and later the Lebesgue integral.

Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données

- Johann Peter Gustav Lejeune Dirichlet (1829, expanded German edition in 1837)

Description: In his habilitation thesis on Fourier series, Riemann characterized this work of Dirichlet as the "*the first profound paper about the subject*".[33] This paper gave the first rigorous proof of the convergence of Fourier series under fairly general conditions (piecewise continuity and monotonicity) by considering partial sums, which Dirichlet transformed into a particular Dirichlet integral involving what is now called the Dirichlet kernel. This paper introduced the nowhere continuous Dirichlet function and an early version of the Riemann-Lebesgue lemma.[34]

On convergence and growth of partial sums of Fourier series

- Lennart Carleson (1966)

Description: Settled Lusin's conjecture that the Fourier expansion of any L^2 function converges almost everywhere.

Geometry

Baudhayana Sulba Sutra

- Baudhayana

Description: Written around the 8th century BC, this is one of the oldest geometrical texts. It laid the foundations of Indian mathematics and was influential in South Asia and its surrounding regions, and perhaps even Greece. Among the important geometrical discoveries included in this text are: the earliest list of Pythagorean triples discovered algebraically, the earliest statement of the Pythagorean theorem, geometric solutions of linear equations, several approximations of π, the first use of irrational numbers, and an accurate computation of the square root of 2, correct to a remarkable five decimal places. Though this was primarily a geometrical text, it also contained some important algebraic developments, including the earliest use of quadratic equations of the forms $ax^2 = c$ and $ax^2 + bx = c$, and integral solutions of simultaneous Diophantine equations with up to four unknowns.

Euclid's Elements

- Euclid

Publication data: c. 300 BC

Online version: Interactive Java version [37]

Description: This is often regarded as not only the most important work in geometry but one of the most important works in mathematics. It contains many important results in geometry, number theory and the first algorithm as well. More than any specific result in the publication, it seems that the major achievement of this publication is the popularization of logic and mathematical proof as a method of solving problems.

The Nine Chapters on the Mathematical Art

- Unknown author

Description: This was a Chinese mathematics book, mostly geometric, composed during the Han Dynasty, perhaps as early as 200 BC. It remained the most important textbook in China and East Asia for over a thousand years, similar to the position of Euclid's *Elements* in Europe. Among its contents: Linear problems solved using the principle known later in the West as the *rule of false position*. Problems with several unknowns, solved by a principle similar to Gaussian elimination. Problems involving the principle known in the West as the Pythagorean theorem. The earliest solution of a matrix using a method equivalent to the modern method.

The Conics

- Apollonius of Perga

Description: The Conics was written by Apollonius of Perga, a Greek mathematician. His innovative methodology and terminology, especially in the field of conics, influenced many later scholars including Ptolemy, Francesco Maurolico, Isaac Newton, and René Descartes. It was Apollonius who gave the ellipse, the parabola, and the hyperbola the names by which we know them.

La Géométrie

- René Descartes

Description: La Géométrie was published in 1637 and written by René Descartes. The book was influential in developing the Cartesian coordinate system and specifically discussed the representation of points of a plane, via real numbers; and the representation of curves, via equations.

Grundlagen der Geometrie

- David Hilbert

Publication data: Hilbert, David (1899). *Grundlagen der Geometrie*. Teubner-Verlag Leipzig.

Description: Hilbert's axiomatization of geometry, whose primary influence was in its pioneering approach to metamathematical questions including the use of models to prove axiom independence and the importance of establishing the consistency and completeness of an axiomatic system.

Regular Polytopes

- H.S.M. Coxeter

Description: *Regular Polytopes* is a comprehensive survey of the geometry of regular polytopes, the generalisation of regular polygons and regular polyhedra to higher dimensions. Originating with an essay entitled *Dimensional Analogy* written in 1923, the first edition of the book took Coxeter 24 years to complete. Originally written in 1947, the book was updated and republished in 1963 and 1973.

Differential geometry

Recherches sur la courbure des surfaces

- Leonard Euler (1760)

Publication data: Mémoires de l'académie des sciences de Berlin **16** (1760) pp. 119–143; published 1767. (Full text [38] and an English translation available from the Dartmouth Euler archive.)

Description: Established the theory of surfaces, and introduced the idea of principal curvatures, laying the foundation for subsequent developments in the differential geometry of surfaces.

Disquisitiones generales circa superficies curvas

- Carl Friedrich Gauss (1827)

Publication data: "Disquisitiones generales circa superficies curvas" [39], *Commentationes Societatis Regiae Scientiarum Gottingesis Recentiores* Vol. **VI** (1827), pp. 99–146; "General Investigations of Curved Surfaces [40]" (published 1965) Raven Press, New York, translated by A.M.Hiltebeitel and J.C.Morehead.

Description: Groundbreaking work in differential geometry, introducing the notion of Gaussian curvature and Gauss' celebrated Theorema Egregium.

Über die Hypothesen, welche der Geometrie zu Grunde Liegen

- Bernhard Riemann (1854)

Publication data: "Über die Hypothesen, welche der Geometrie zu Grunde Liegen" [41], *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 13, 1867.

Description: Riemann's famous Habiltationsvortrag, in which he introduced the notions of a manifold, Riemannian metric, and curvature tensor.

Leçons sur la théorie génerale des surfaces

- Gaston Darboux

Publication data: Darboux, Gaston (1887,1889,1896). *Leçons sur la théorie génerale des surfaces: Volume I* [42]*, Volume II* [43]*, Volume III* [44]*, Volume IV* [44]. Gauthier-Villars.

Description: A treatise covering virtually every aspect of the 19th century differential geometry of surfaces.

→ Topology

Analysis situs

- Henri Poincaré (1895, 1899-1905)

Description: Poincaré's Analysis situs and his Compléments à l'Analysis Situs laid the general foundations for → algebraic topology. In these papers, Poincaré introduced the notions of homology and the → fundamental group, provided an early formulation of → Poincaré duality, gave the Euler-Poincaré characteristic for chain complexes, and mentioned several important conjectures including the Poincaré conjecture.

Grundzüge der Mengenlehre

- Felix Hausdorff (1914)
- Reprinted and commented in: *Gesammelte Werke Bd. 2* / Herausgegeben von E. Brieskorn, S.D.Chatterji *et al.*. – Berlin, 2002

Description: This book founded (general) topology by giving the axioms for a (Hausdorff) topological space.

L'anneau d'homologie d'une représentation, *Structure de l'anneau d'homologie d'une représentation*

- Jean Leray (1946)

Description: These two Comptes Rendus notes of Leray from 1946 introduced the novel concepts of sheafs, sheaf cohomology, and spectral sequences, which he had developed during his years of captivity as a prisoner of war. Leray's announcements and applications (published in other Comptes Rendus notes from 1946) drew immediate attention from other mathematicians. Subsequent clarification, development, and generalization by → Henri Cartan, Jean-Louis Koszul, Armand Borel, → Jean-Pierre Serre, and Leray himself allowed these concepts to be understood

and applied to many other areas of mathematics.[35] Dieudonné would later write that these notions created by Leray "*undoubtedly rank at the same level in the history of mathematics as the methods invented by Poincaré and Brouwer*". [9]

Quelques propriétés globales des variétés differentiables

- → René Thom (1954)

Description: In this paper, Thom proved the Thom transversality theorem, introduced the notions of oriented and unoriented cobordism, and demonstrated that cobordism groups could be computed as the homotopy groups of certain Thom spaces. Thom completely characterized the unoriented cobordism ring and achieved strong results for several problems, including Steenrod's problem on the realization of cycles.[9] [36]

Category theory

General theory of natural equivalences

- → Samuel Eilenberg and → Saunders Mac Lane (1945)

Description: The first paper on category theory. Mac Lane later wrote in *Categories for the Working Mathematician* that he and Eilenberg introduced categories so that they could introduce functors, and they introduced functors so that they could introduce natural equivalences. Prior to this paper, "natural" was used in an informal and imprecise way to designate constructions that could be made without making any choices. Afterwards, "natural" had a precise meaning which occurred in a wide variety of contexts and had powerful and important consequences.

Categories for the Working Mathematician

- → Saunders Mac Lane (1971, second edition 1998)

Description: Saunders Mac Lane, one of the founders of category theory, wrote this exposition to bring categories to the masses. Mac Lane does not get lost in pointless abstraction, but instead brings to the fore the important concepts that make category theory useful, such as adjoint functors and universal properties. His text is more comprehensive than most mathematicians will ever need, and consequently is also an excellent reference.

Set theory

Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen

- Georg Cantor (1874)

Description: Contains the first proof that the set of all real numbers is uncountable; also contains a proof that the set of algebraic numbers is denumerable.

Grundzüge der Mengenlehre

- Felix Hausdorff

Description: First published in 1914, this was the first comprehensive introduction to set theory. Besides the systematic treatment of known results in set theory, the book also contains chapters on measure theory and topology, which were then still considered parts of set theory. Here Hausdorff presents and develops highly original material which was later to become the basis for those areas.

The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory

- Kurt Gödel (1938)

Description: Gödel proves the results of the title. Also, in the process, introduces the class L of constructible sets, a major influence in the development of axiomatic set theory.

The Independence of the Continuum Hypothesis

- Paul J. Cohen (1963, 1964)

Description: Cohen's breakthrough work proved the independence of the continuum hypothesis and axiom of choice with respect to Zermelo-Fraenkel set theory. In proving this Cohen introduced the concept of *forcing* which led to many other major results in axiomatic set theory.

Logic

Begriffsschrift

- Gottlob Frege (1879)

Description: Published in 1879, the title ***Begriffsschrift*** is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "*a formula language, modelled on that of arithmetic, of pure thought*". Frege's motivation for developing his formal logical system was similar to Leibniz's desire for a *calculus ratiocinator*. Frege defines a logical calculus to support his research in the foundations of mathematics. ***Begriffsschrift*** is both the name of the book and the calculus defined therein. It was arguably the most significant publication in logic since Aristotle.

Formulario mathematico

- Giuseppe Peano (1895)

Description: First published in 1895, the **Formulario mathematico** was the first mathematical book written entirely in a formalized language. It contained a description of mathematical logic and many important theorems in other branches of mathematics. Many of the notations introduced in the book are now in common use.

Principia Mathematica

- Bertrand Russell and Alfred North Whitehead (1910-1913)

Description: The ***Principia Mathematica*** is a three-volume work on the foundations of mathematics, written by Bertrand Russell and Alfred North Whitehead and published in 1910-1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The questions remained whether a contradiction could be derived from the Principia's axioms, and whether there exists a mathematical statement which could neither be proven nor disproven in the system. These questions were settled, in a rather surprising way, by Gödel's incompleteness theorem in 1931.

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I

- Kurt Gödel (1931)

Online version: Online version [47]

Description: In mathematical logic, **Gödel's incompleteness theorems** are two celebrated theorems proved by Kurt Gödel in 1931. The first incompleteness theorem states:

> For any formal system such that (1) it is ω-consistent (omega-consistent), (2) it has a recursively definable set of axioms and rules of derivation, and (3) every recursive relation of natural numbers is definable in it, there exists a formula of the system such that, according to the intended interpretation of the system, it expresses a truth about natural numbers and yet it is not a theorem of the system.

Combinatorics

On sets of integers containing no k elements in arithmetic progression

- Endre Szemerédi (1975)

Description: Settled a conjecture of Paul Erdős and Paul Turán that if a sequence of natural numbers has positive upper density then it contains arbitrarily long arithmetic progressions. Szemerédi's solution has been described as a "masterpiece of combinatorics"[37] and it introduced new ideas and tools to the field including the Szemerédi regularity lemma.

Graph theory

Solutio problematis ad geometriam situs pertinentis

- Leonhard Euler (1741)
- Euler's original publication [49] (in Latin)

Description: Euler's solution of the Königsberg bridge problem in *Solutio problematis ad geometriam situs pertinentis* (*The solution of a problem relating to the geometry of position*) is considered to be the first theorem of graph theory.

On the evolution of random graphs

- Paul Erdös and Alfréd Rényi (1960)

Description: Provides a detailed discussion of sparse random graphs, including distribution of components, occurrence of small subgraphs, and phase transitions.[38]

Network Flows and General Matchings

- Ford, L., & Fulkerson, D.
- Flows in Networks. Prentice-Hall, 1962.

Description: Ford and Fulkerson paper on Network Flows. The algorithm along with many ideas on flow-based models can be found in their book.

Complexity theory

See List of important publications in computer science.

Probability theory

See list of important publications in statistics.

Game theory

Zur Theorie der Gesellschaftsspiele

- John von Neumann (1928)

Description: Went well beyond Émile Borel's initial investigations into strategic two-person game theory by proving the minimax theorem for two-person, zero-sum games.

Theory of Games and Economic Behavior

- Oskar Morgenstern, John von Neumann (1944)

Description: This book led to the investigation of modern game theory as a prominent branch of mathematics. This profound work contained the method for finding optimal solutions for two-person zero-sum games.

Equilibrium Points in N-person Games

- John Forbes Nash
- *Proceedings of the National Academy of Sciences* 36 (1950), 48–49. MR0031701 [51]
- "Equilibrium Points in N-person Games" [52]

Description: Nash equilibrium

On Numbers and Games

- John Horton Conway

Description: The book is in two, {0,1|}, parts. The zeroth part is about numbers, the first part about games - both the values of games and also some real games that can be played such as Nim, Hackenbush, Col and Snort amongst the many described.

Winning Ways for your Mathematical Plays

- Elwyn Berlekamp, John Conway and Richard K. Guy

Description: A compendium of information on mathematical games. It was first published in 1982 in two volumes, one focusing on Combinatorial game theory and surreal numbers, and the other concentrating on a number of specific games.

Fractals

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

- Benoît Mandelbrot

Description: A discussion of self-similar curves that have fractional dimensions between 1 and 2. These curves are examples of fractals, although Mandelbrot does not use this term in the paper, as he did not coin it until 1975. Shows Mandelbrot's early thinking on fractals, and is an example of the linking of mathematical objects with natural forms that was a theme of much of his later work.

Numerical analysis

Numerical linear algebra

The Algebraic Eigenvalue Problem

- James H. Wilkinson (1965)

Description:

Optimization

Method of Fluxions

- Isaac Newton

Description: *Method of Fluxions* was a book written by Isaac Newton. The book was completed in 1671, and published in 1736. Within this book, Newton describes a method (the Newton-Raphson method) for finding the real zeroes of a function.

Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies

- Joseph Louis Lagrange (1761)

Description: Major early work on the calculus of variations, building upon some of Lagrange's prior investigations as well as those of Euler. Contains investigations of minimal surface determination as well as the initial appearance of Lagrange multipliers.

The New Variational Method

- Leonid Kantorovich
- This work was published in 1930's in the erstwhile Soviet Republic.

Description: Kantorovich wrote the first paper on production planning, which used Linear Programs as the model. He proposed the simplex algorithm as a systematic procedure to solve these Linear Programs. He received the Nobel prize for this work in 1975.

Decomposition Principle for Linear Programs

- George Dantzig and P. Wolfe
- Operations Research 8:101–111, 1960.

Description: Dantzig's is considered the father of Linear Programming in the western world. He independently invented the simplex algorithm. Dantzig and Wolfe worked on decomposition algorithms for large scale linear programs in factory and production planning.

How good is the simplex algorithm?

- Victor Klee and George J. Minty
- In: O. Shisha (ed.) Inequalities III, Academic Press (1972) 159–175.

Description: Klee and Minty gave an example showing that the simplex method can take exponentially many steps to solve a linear program if it chooses the greedy ascent rule.

Linear Programming and Polynomial time algorithms

- L. Khachiyan
- Doklady Akademii Nauk SSSR 244 (1979) pp. 1093–1096 (Russian).

Description: Khachiyan's work on Ellipsoid method. This was the first polynomial time algorithm for Linear programming.

New polynomial-time algorithm for linear programming

- Karmarkar, N.
- Combinatorica 4, 373–395, 1984.

Description: Karmarkars path-breaking work on Interior-Point algorithms for Linear Programming.

Interior Point Polynomial Algorithms in Convex Programming

- Yurii Nesterov and A. Nemirovski
- Philadelphia : Society for Industrial and Applied Mathematics, 1994. (SIAM Studies in Applied Mathematics).

Description: Nesterov and Nemirovski's work on self-concordant barriers and interior-point methods for general convex programming. All their series of papers (both individual and combined) is compiled more coherently in the following "bible" of convex optimization.

Early manuscripts

These are publications that are not necessarily relevant to a mathematician nowadays, but are nonetheless important publications in the history of mathematics.

Rhind Mathematical Papyrus

- Ahmes (scribe)

Description: It is one of the oldest mathematical texts, dating to the Second Intermediate Period of ancient Egypt. It was copied by the scribe Ahmes (properly *Ahmose*) from an older Middle Kingdom papyrus. It laid the foundations of Egyptian mathematics and in turn, later influenced Greek and Hellenistic mathematics. Besides describing how to obtain an approximation of π only missing the mark by less than one per cent, it is describes one of the earliest attempts at squaring the circle and in the process provides persuasive evidence against the theory that the Egyptians deliberately built their pyramids to enshrine the value of π in the proportions. Even though it would be a strong overstatement to suggest that the papyrus represents even rudimentary attempts at analytical geometry, Ahmes did make use of a kind of an analogue of the cotangent.

Archimedes Palimpsest

- Archimedes of Syracuse

Description: Although the only mathematical tools at its author's disposal were what we might now consider secondary-school geometry, he used those methods with rare brilliance, explicitly using infinitesimals to solve problems that would now be treated by integral calculus. Among those problems were that of the center of gravity of a solid hemisphere, that of the center of gravity of a frustum of a circular paraboloid, and that of the area of a region bounded by a parabola and one of its secant lines. For explicit details of the method used, see Archimedes' use of infinitesimals.

The Sand Reckoner

- Archimedes of Syracuse

Online version: Online version [53]

Description: The first known (European) system of number-naming that can be expanded beyond the needs of everyday life.

Textbooks

Synopsis of Pure Mathematics

- G. S. Carr

Description: Contains over 6000 theorems of mathematics, assembled by George Shoobridge Carr for the purpose of training students in the art of mathematics, studied extensively by Ramanujan. (first half here) [54] It was one of the few books that attempts to summarize the entirety of known mathematics.

Arithmetick: or, The Grounde of Arts

- Robert Recorde

Description: Written in 1542, it was the first really popular arithmetic book written in the English Language.

Cocker's Arithmetick

- Edward Cocker (authorship disputed)

Description: Textbook of arithmetic published in 1678 by John Hawkins, who claimed to have edited manuscripts left by Edward Cocker, who had died in 1676. This influential mathematics textbook used to teach arithmetic in schools in the United Kingdom for over 150 years.

The Schoolmaster's Assistant, Being a Compendium of Arithmetic both Practical and Theoretical

- Thomas Dilworth

Description: An early and popular English arithmetic textbook published in America in the eighteenth century. The book reached from the introductory topics to the advanced in five sections.

Course of Pure Mathematics

- G. H. Hardy

Description: A classic textbook in introductory mathematical analysis, written by G. H. Hardy. It was first published in 1908, and went through many editions. It was intended to help reform mathematics teaching in the UK, and more specifically in the University of Cambridge, and in schools preparing pupils to study mathematics at Cambridge. As such, it was aimed directly at "scholarship level" students — the top 10% to 20% by ability. The book contains a large number of difficult problems. The content covers introductory calculus and the theory of infinite series.

Moderne Algebra

- B. L. van der Waerden

Description: The first introductory textbook (graduate level) expounding the abstract approach to algebra developed by Emil Artin and Emmy Noether. First published in German in 1931 by Springer Verlag. A later English translation was published in 1949 by Frederick Ungar Publishing Company.

Algebra

- → Saunders Mac Lane and Garrett Birkhoff

Description: A definitive introductory text for abstract algebra using a category theoretic approach. Both a rigorous introduction from first principles, and a reasonably comprehensive survey of the field.

Algebraic Geometry

- Robin Hartshorne

Description: The first comprehensive introductory (graduate level) text in algebraic geometry that used the language of schemes and cohomology. Published in 1977, it lacks aspects of the scheme language which are nowadays considered central, like the functor of points.

Naive Set Theory

- Paul Halmos

Description: An undergraduate introduction to not-very-naive set theory which has lasted for decades. It is still considered by many to be the best introduction to set theory for beginners. While the title states that it is naive, which is usually taken to mean without axioms, the book does introduce all the axioms of Zermelo-Fraenkel set theory and gives correct and rigorous definitions for basic objects. Where it differs from a "true" axiomatic set theory book is its character: There are no long-winded discussions of axiomatic minutiae, and there is next to nothing about advanced topics like large cardinals. Instead it tried, and succeeds, in being intelligible to someone who has never thought about set theory before.

Cardinal and Ordinal Numbers

- Waclaw Sierpinski

Description: The *nec plus ultra* reference for basic facts about cardinal and ordinal numbers. If you have a question about the cardinality of sets occurring in everyday mathematics, the first place to look is this book, first published in the early 1950s but based on the author's lectures on the subject over the preceding 40 years.

Set Theory: An Introduction to Independence Proofs

- Kenneth Kunen

Description: This book is not really for beginners, but graduate students with some minimal experience in set theory and formal logic will find it a valuable self-teaching tool, particularly in regard to forcing. It is far easier to read than a true reference work such as Jech, *Set Theory*. It may be the best textbook from which to learn forcing, though it has the disadvantage that the exposition of forcing relies somewhat on the earlier presentation of Martin's axiom.

Topologie

- Pavel Sergeevich Alexandrov
- → Heinz Hopf

Description: First published round 1935, this text was a pioneering "reference" text book in topology, already incorporating many modern concepts from set-theoretic topology, homological algebra and homotopy theory.

General Topology

- John L. Kelley

Description: First published in the mid-1950s,for many years the only introductory graduate level textbook in the U.S.A. teaching the basics of point set, as opposed to algebraic, topology. Prior to this the material, essential for advanced study in many fields, was only available in bits and pieces from texts on other topics or journal articles.

Topology from the Differentiable Viewpoint

- John Milnor

Description: This short book introduces the main concepts of differential topology in Milnor's lucid and concise style. While the book does not cover very much, its topics are explained beautifully in a way that illuminates all their details.

Number Theory, An approach through history from Hammurapi to Legendre

- André Weil

Description: An historical study of number theory, written by one of the 20th century's greatest researchers in the field. The book covers some thirty six centuries of arithmetical work but the bulk of it is devoted to a detailed study and exposition of the work of Fermat, Euler, Lagrange, and Legendre. The author wishes to take the reader into the workshop of his subjects to share their successes and failures. A rare opportunity to see the historical development of a subject through the mind of one of its greatest practitioners.

An Introduction to the Theory of Numbers

- G. H. Hardy and E. M. Wright

Description: This book was first published in 1938, and is still in print, with the latest edition being the 6th (2008). It is likely that almost every serious student and researcher into number theory has consulted this book, and probably has it on their bookshelf. It was not intended to be a textbook, and is rather an introduction to a wide range of differing areas of number theory which would now almost certainly be covered in separate volumes. The writing style has long been regarded as exemplary, and the approach gives insight into a variety of areas without requiring much more than a good grounding in algebra, calculus and complex numbers.

Popular writing

Gödel, Escher, Bach

- Douglas Hofstadter

Description: Gödel, Escher, Bach: an Eternal Golden Braid is a Pulitzer Prize-winning book, first published in 1979 by Basic Books. It is a book about how the creative achievements of logician Kurt Gödel, artist M. C. Escher and composer Johann Sebastian Bach interweave. As the author states: "I realized that to me, Gödel and Escher and Bach were only shadows cast in different directions by some central solid essence. I tried to reconstruct the central object, and came up with this book."

The World of Mathematics

- James R. Newman

Description: The World of Mathematics was specially designed to make mathematics more accessible to the inexperienced. It comprises nontechnical essays on every aspect of the vast subject, including articles by and about scores of eminent mathematicians, as well as literary figures, economists, biologists, and many other eminent thinkers. Includes the work of Archimedes, Galileo, Descartes, Newton, Gregor Mendel, Edmund Halley, Jonathan Swift, John Maynard Keynes, Henri Poincaré, Lewis Carroll, George Boole, Bertrand Russell, Alfred North Whitehead, John von Neumann, and many others. In addition, an informative commentary by distinguished scholar James R. Newman precedes each essay or group of essays, explaining their relevance and context in the history and development of mathematics. Originally published in 1956, it does not include many of the exciting discoveries of the later years of the 20th century but it has no equal as a general historical survey of important topics and applications.

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